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Business Information Systems

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OPERATIONS RESEARCH

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Contents

Chapter 1

Introduction

1.1 THE ORIGINS AND THE NATURE OF OPERATIONS RESEARCH

During World War II scientists and engineers were asked to analyse several military problems: developing effective methods of using the newly invented radar, how to better manage convoy and antisubmarine, bombing and military operations. The applications of mathematics and scientific methods to military operations was called Operations Research (-commonly referred to as OR). The term OR (or often Management Science) means a scientific approach to decision making which seeks to determine how best design and coordinate a system usually under conditions requiring the allocation of the scarce resources. The methodology of the operations research is employed to problems that concern how to conduct and coordinate the operations (i.e. activities) within organizations. The nature of organization is essentially immaterial, OR has been applied extensively in such diverse areas as:

- Manufacturing

- Transportation

- Telecommunication

- Financial planning

- Public services

- Health care

- The military

The research part of the name of OR means that scientific methods are used to investigate the problem considered. The process begins by carefully observing and formulating the problem, including gathering all the relevant data. The second step is to construct a scientific (typically mathematical) model that attempts to abstract the essence of the real problem. It is then assumed (hypothesized) that the model is a sufficiently precise representation of the essential features of the situation for the conclusions (solutions) obtained from the model to also be valid for the real problem. Next, suitable experiments are conducted to test this hypothesis, modify it as needed, and eventually verify some form of the hypothesis (This step is frequently referred to as model validation).

But OR is also concerned with the practical management of the organization. Therefore, to be successful, OR must also provide positive, understandable conclusions to the decision maker(s) when they are needed.

An additional characteristic is that OR frequently attempts to find a best solution(referred to as an optimal solution) for the problem under consideration (There may be more than one the best solution, in which case we choose one of them). Rather than simply improving the status quo, the aim in OR is to identify a best possible course of action. This search for optimality is an important theme in OR. Another important characteristic of OR is the so called team approach. It is evident that no single individual should be expected to be expert on all the aspects of OR work or the problems typically considered: this would require a group of people having diverse backgrounds and skills. Such an OR team needs to include people who collectively are highly trained in mathematics, statistics and probability theory, economics, business administration, computer sciences, engineering, the behavioral sciences and the special techniques of OR. There are a lot of techniques to solve mathematical problems that can arise in practice. The most prominent is linear programming. Other techniques include integer programming, dynamic programming, networks programming (in which the problem can be modeled as a network), nonlinear programming. Another group consists of queuing and simulations models. In practice, the algorithms (methods of solutions for the each kind of model) are executed by commercial software packages.

OR has had an impressive impact on improving the efficiency of numerous organizations around the world, and has made a significant contribution to increasing the productivity of the economies of various countries. There are now a lot of countries in the International Federation of Operations Research Societies (IFORS) with each country having a national OR society (for example BOS in Poland).

1.2 OVERVIEW OF THE OR MODELLING APPROACH

OR is applied to problems that concern how to conduct and coordinate operations(i.e. activities) within an organization. The major phases of a typical OR study are the following:

1. Define the problem and gather relevant data.

2. Formulate a mathematical model to represent the problem.

3. Develop a computer-based procedure for deriving a solution to the problem resulting from the model.

4. Test the model and refine it as needed.

5. Prepare for the ongoing application of the model as prescribed by management.

6. Implement.

Now we will discuss these phases in turn starting with the process of defining the problem and gathering data which includes determining such things as:

- the appropriate objectives,

- constraints on what can be done,

- interrelations between the area to be studied and the other areas of organization,

- possible alternative courses of action,

- time limits for making a decision

The second phase is to formulate a mathematical model. A mathematical model is defined by a system of equations and related mathematical expressions that describe the essence of a problem. A main elements of a model are the following:

Decision variables If there are n related quantifiable decisions to be made, they are represented as decision variables $x_1, x_2, \ldots, x_n$ whose values are to be determined.

Objective function The appropriate (overall) measure of performance(e.g. profit) is expressed as a mathematical function of the decision variables (for example $P = 5x_1 + 2x_2 + \cdots + 20x_n$).

Constraints Any restrictions on the values that can be assigned to the decision variables are expressed mathematically, typically by means of inequalities or equations (for example $(x_1)^2 - 2x_1x_2 + (x_2)^2 \leq 25$).

Parameters of the model The constants (namely, the coefficients and right- hand sides) in the constraints and the objective function.

The next phase is to solve the resulting model. **The problem is to choose the values of the decision variables so as to optimize (maximize or minimize) the objective function, subject to the specific constraints.** An important part of the model-building process is determining the appropriate values to assign to the parameters of the model. This requires **gathering relevant data**. Another important part of model solution is so called **Sensitivity analysis**. This aims to answer the following question: How would the solution derived from the model change (if at all) if the values assigned to the parameters were changed to other plausible values (There is uncertainty about true values of the parameters).

Deriving the solution from the model

- Develop a computer-based procedure for deriving the solution to the problem from the model-sometimes it possible to use standard OR algorithms (there are a number of readily available software pacakages).

- Search for **an optimal**, or best, **solution**. These solutions are optimal only with respect to the model being used (since the model is an idealized, rather then an exact, representation of the real problem).

- Use heuristic procedures (i.e. intuitively designed procedures that do not guarantee an optimal solution) to find a good **suboptimal solution** in case where the time or cost required to find an optimal solution to the model is very large.

- **Postoptimality analysis** or **What-if analysis**. This answers the question - what would happen to the optimal solution if different assumptions are made about future conditions? It also involves conducting **sensitivity analysis** to determine which parameters of the model are most critical in determining the solution.

- Test the model and refine it as needed.

In this textbook, we concentrate on formulating mathematical models commonly used in modelling decision makers' problems and solving the mathematical models. The main subjects of the textbook are following:

- Linear Programming

- – Introduction to Linear Programming
- – Solving Linear Programming Problems - The Simplex Method
- – Duality Theory and Sensitivity Analysis

- Integer Programming

- Decision Analysis

- Multiple Objective Linear Programming and Goal Programming

This course is based on the following textbooks:

- Hiller F. S. Hillier, G.J. Liberman: *Introduction to Operations Research*, 7th ed., McGrawHill,2001.

- Winston W.L. *Operations Research: Applications and Algorithms*, PWS-KENT Publishing Company, Boston, 1987.

Additional literature

- Anderson D.R.,D.J.Sweeney, and T.A.Williams: *An Introduction to Management Science*, 8th ed., West,St.Paul,Mn,2000.

- Taha, H. *Operations Research: An Introduction*, 8th ed., Pearson Prentice Hall, Upper Sadle River,NJ,2007.

- Williams H.P.: *Model Building in Mathematical Programming*, 3d ed.,Wiley, New York,1990.

Chapter 2

Introduction to Linear Programming

2.1 FORMULATION OF LINEAR PROGRAM-MING PROBLEMS

Operations Research (OR) means a scientific approach to decision making, which seeks to determine how to optimally design and operate a system, usually under conditions requiring the allocation of scarce resources.

An OR study has the following major phases:

| I. Define the problem and gather relevant data. | $\Rightarrow$

| II. Formulate a mathematical model to represent the problem | $\Rightarrow$

| III. Develop a procedure for deriving a solution to the problem | $\Rightarrow$

| IV. Test the model and refine it as needed | $\Rightarrow$

| V. Implement |

Some of these phases will be illustrated the following example.

Prototype Example :

I. The GW firm manufactures two types of products W_1 and W_2 using two types of raw materials S_1 and S_2. The GW firm wishes to determine how many products of each type to produce, so as to achieve the highest possible profit.

II. The following data are available:

- 1 unit of W_1 requires 2 kg of S_1 and 1 kg of S_2.

- 1 unit of W_2 requires 1 kg of S_1 and 1 kg of S_2.

- At present, GW has 100 kg of S_1 and 80 kg of S_2 in stock.

- Each unit of W_1 gives a profit of \$3 and each unit of W_2 gives a profit of \$2.

- Demand for W_1 equals 40 units and demand for W_2 is unlimited.

III. We define ***the decision variables***: x_1 - amount of product W_1 produced, x_2 - amount of product W_2 produced, Z - total profit from producing W_1 and W_2.

The mathematical model is as follows:

$$
\begin{aligned}
&\max Z = 3x_1 + 2x_2 && \text{[Maximize profit]} \\
&2x_1 + x_2 \leq 100 && \text{[Constraint on raw material } S_1 \text{]} \\
&x_1 + x_2 \leq 80 && \text{[Constraint on raw material } S_2 \text{]} \\
&x_1 \leq 40 && \text{[Demand for } W_1 \text{]} \\
&x_1 \geq 0 && \text{[Sign restriction]} \\
&x_2 \geq 0 && \text{[Sign restriction]}
\end{aligned}
$$

The problem is to choose values of x_1 and x_2, so as to maximize $Z = 3x_1 + 2x_2$ subject to the restrictions. We want to derive a production plan that maximizes GW's profit without violating the constraints on limited resources.

IV. We use an algorithm (the next part of the course will focus on the simplex method for solving linear programming problems). We obtain: $x_1 = 20$, $x_2 = 60$, $Z = 180$, so GW should produce 20 units of W_1 and 60 units of W_2 with a resulting total profit of \$180.

2.2 THE LINEAR PROGRAMMING MODEL

The mathematical model for a general mathematical programming problem is as follows:

$$
\begin{aligned}
&\max(\min)z = f(x_1, \ldots, x_n) && \text{[Objective function]} \\
&g_1(x_1, \ldots, x_n) \leq (\geq, =)b_1 && \text{[Constraint 1]} \\
&\ldots \\
&g_m(x_1, \ldots, x_n) \leq (\geq, =)b_m && \text{[Constraint } m \text{]}
\end{aligned}
$$

$x_1, \ldots, x_n$ are called **decision variables** (They should completely describe the decision to be made). A solution for which all the constraints are satisfied is called **a feasible solution**. A feasible solution with the largest (smallest) value of the objective function is called **an optimal solution**.

The function $h(x_1, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ of $x_1, x_2, \ldots, x_n$ is **a linear function**. A model in which all the functions $f, g_1, \ldots, g_m$ are linear is called **a linear model** or **a linear programming problem**. A linear programming model(LPM) is defined as follows:

$$
\begin{aligned}
&\max(\min)z = c_1x_1 + c_2x_2 + \cdots + c_nx_n && \text{[Objective function]} \\
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq (\geq, =)b_1 && \text{[Constraint 1]} \\
&\ldots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq (\geq, =)b_m && \text{[Constraint } m \text{]} \\
&x_1 \geq 0, \ldots, x_r \geq 0, \ \ r \leq n && \text{[Nonnegativity constraints]}
\end{aligned}
$$

The sign constraints $x_i \geq 0$ are special linear constraints (nonnegativity constraints).

2.3 EXAMPLES OF LINEAR PROGRAMMING PROBLEMS

1. *The Diet Problem.* Person X requires that all the food she/he eats come from one of the four "basic food groups": butter, bread, orange juice and ham. She/he wants to prepare breakfast only from these foods. Each scoop of butter costs\$5, each slice (piece) of bread costs\$2, each bottle of orange juice

costs \$3, and each piece (slice) of ham costs \$8. Each day X must ingest at least 500 calories, 6 oz of sugar, 10 oz of carbohydrates, and 8 oz of fat. The nutritional content per unit of each food is shown in the following table:

	Calories	**Sugar**	**Carbohydrates**	**Fat**	Unit price(\$)
Butter (1 scoop)	400	3	2	2	5
Bread (1 piece)	200	2	2	4	2
Orange juice (1 bottle)	150	0	4	1	3
Ham (1 slice)	500	0	4	5	8

Formulate a linear programming model that satisfies the daily nutritional requirements of person X at minimum cost.

Decision variables:

- x_1 - amount(number of scoops) of butter,

- x_2 - amount(number of pieces) of bread,

- x_3 - number of bottles of orange juice,

- x_4 - amount (number of slices) of ham.

Linear Programming Model:

$$\begin{aligned}
&\min Z = 5x_1 + 2x_2 + 3x_3 + 8x_4 && \text{[Minimize total cost]}\\
&400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 && \text{[Calorie constraint]}\\
&3x_1 + 2x_2 \geq 6 && \text{[Sugar constraint]}\\
&2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 && \text{[Carbohydrates constraint]}\\
&2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 && \text{[Fat constraint]}\\
&x_i \geq 0, \; i = 1, \ldots, 4 && \text{[Sign constraint]}
\end{aligned}$$

The optimal solution to this problem (solved by computer) is $x_1 = 0$, $x_2 = 3$, $x_3 = 1$, $x_4 = 0$, $Z = 9$. Thus, the minimum-cost diet incurs a cost of \$9 and consists of 3 pieces of bread and one bottle of orange juice.

2. *Production Process Model* (W.L. Winston Operations Research: Applications and Algorithms, Example 3.11, pp.87-90) Rylon Corporation manufactures four types of perfumes: Brute, Chanelle, Super Brute and Super Chanelle. The production process is given in the Figure below.
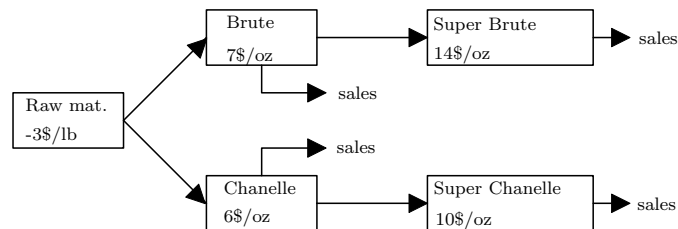
- Processing 1 lb of raw material requires 1 hour of working time and gives 3 oz of Brute and 4 oz of Chanelle.

- 1 oz of Brute requires an additional 3 hours of processing to yield 1 oz of Super Brute.

- 1 oz of Chanelle requires an additional 2 hours of processing to yield 1 oz of Super Chanelle.

- Rylon has 6000 hours of time available and can purchase up to 4000 lb of raw material.

R.C must determine how much material to purchase and how much of each type of perfume should be produced and wants to maximize profit(=revenues from perfume sales - processing costs).

Decision variables:

- x_1 - number of ounces of Brute sold,

- x_2 - number of ounces of Super Brute sold,

- x_3 - number of ounces of Chanelle sold,

- x_4 - number of ounces of Super Chanelle sold,

- x_5 - number of pounds(lb) of raw material purchased.

Linear Programming Model:

$$\max z = 7x_1 + 14x_2 + 6x_3 + 10x_4 - 3x_5 \quad \text{[Maximize profit]}$$
$$x_1 + x_2 - 3x_5 = 0 \qquad\qquad \text{[Manufacturing Brute and Super Brute]}$$
$$x_3 + x_4 - 4x_5 = 0 \qquad\qquad \text{[Manufacturing Chanelle and Super Chanelle]}$$
$$x_5 \leq 4000 \qquad\qquad \text{[Limit on raw materials]}$$
$$3x_2 + 2x_4 + x_5 \leq 6000 \qquad\qquad \text{[Limit on working time]}$$
$$x_i \geq 0, \ i = 1, \ldots, 5 \qquad\qquad \text{[Sign restrictions]}$$

After solving the model by computer we obtain the following optimal solution: Z=172 666.667\$ (profit), $x_1 = 11333.333$ oz of Brute, $x_2 = 666.667$ oz of Super Brute, $x_3 = 16000$ oz of Chanelle, $x_4 = 0$ oz of Super Chanelle and $x_5 = 4000$lb of raw material.

3. *An Inventory Model - Multiperiod Decision problems.* A factory must produce a certain product over the next four quarters. The demand for each quarter is known. The factory wants to minimize total costs and has to meet all demands on time. The appropriate data are given in the following table:

	Q I	Q II	Q III	Q IV
Demand (units)	30	60	70	25
Maximum production level(units)	60	60	60	60
Production cost (\$/unit)	55	50	50	55
Storage cost(\$/unit/quarter)	2	2	3	-

Decision variables:

- x_i - number of units produced during quarter i, $i = 1, \ldots 4$.

- m_i - number of units of the product in store (on the inventory) at the end of quarter i, $i = 1, \ldots 4$.
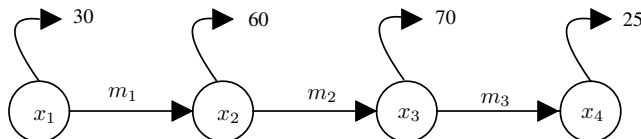
Linear Programming Model:

$\min z = 55x_1 + 50x_2 + 50x_3 + 55x_4 + 2m_1 + 2m_2 + 3m_3$ [Minimum total cost]
$x_1 - m_1 = 30$ [Balance for I]
$m_1 + x_2 - m_2 = 60$ [Balance for II]
$m_2 + x_3 - m_3 = 70$ [Balance for III]
$m_3 + x_4 = 25$ [Balance for IV]
$x_i \le 60, \ i = 1, \dots 4$ [Production capacity]
$x_i \ge 0, \ m_i \ge 0, \ i = 1, \dots, 4$ [Sign restrictions]

The optimal solution to this problem is $Z = 9615$: $x_1 = 40$, $x_2 = 60$, $x_3 = 60$, $x_4 = 25$, $m_1 = 10$, $m_2 = 10$, $m_3 = 0$.

4. *Multiperiod Financial Model.* (W.L. Winston Operations Research: Applications and Algorithms, Example 3.13, pp.95-97) Finco Invest. Corp. must determine an investment strategy for the next three years. At present (time 0), \$100000 is available for investment. Investments A,B,C,D and E are available. The cash flow associated with investing \$1 in each investment is given in the table below.

	0	1	2	3
A	-1\$	+0.5\$	+1\$	-
B	-	-1\$	+0.5\$	1\$
C	-1\$	+1.2\$	-	-
D	-1\$	-	-	+1.9\$
E	-	-	-1\$	+1.5\$

For example, \$1 invested in B requires \$1 in cash at time 1 and returns \$0.5 at time 2 and \$1 at time 3. To ensure that the company's portfolio is diversified, Finco requires that at most \$75000 be placed in any single investment. In addition, Finco can earn interest at 8% per year by keeping uninvested cash in a bank. Returns from investments may be immediately reinvested. Formulate a linear programming problem whose solution will maximize the amount of cash in hand at time 3. Finco must decide how much money should be placed in each investment(including the bank).

Decision variables:

- x_A, x_B, x_C, x_D, x_E - dollars invested in investment A,B,C,D, and E respectively,

- y_0, y_1, y_2 - dollars invested in the bank at time t (t=0,1,2).

The linear programming model:

$\max z = x_B + 1.9x_D + 1.5x_E + 1.08y_2$ [Maximize the amount of cash in hand at time 3]
$x_A + x_C + x_D + y_0 = 100000$ [Balance at time 0]
$0.5x_A + 1.2x_C + 1.08y_0 - x_B - y_1 = 0$ [Balance at time 1]
$x_A + 0.5x_B + 1.08y_1 - x_E - y_2 = 0$ [Balance at time 2]
$x_A, x_B, x_C, x_D, x_E \le 75000$ [Limit on dollars invested in investments]
$x_A, \dots, x_E, y_0, y_1, y_2 \ge 0$ [Nonnegativity constraints]

The optimal solution is $Z = 218500$, $x_A = 60000$, $x_B = 30000$, $x_D = 40000$, $x_E = 75000$, $x_C = y_0 = y_1 = y_2 = 0$. Finco should not invest in C nor in the bank.

2.4   THE GRAPHICAL METHOD FOR SOLVING LPM.

This method can be used when the number of variables is at most three.

Example 1. Solve the following linear programming problem:

$$\max Z = 3x_1 + 2x_2$$
$$2x_1 + x_2 \leq 100$$
$$x_1 + x_2 \leq 80$$
$$x_1 \leq 40$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$
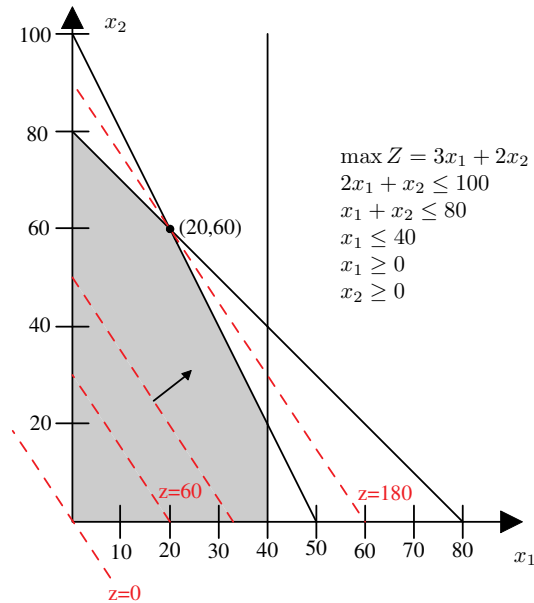
Step 1 We label the variables x_1, x_2 and coordinate axes x_1, x_2 and graph the set of points satisfying the linear inequalities (or equalities) involving the two variables (it is possible to do this in three dimensional space). This set of points is called ***the feasible region*** (the shaded area on the Figure).

Step 2 We draw contours of the objective function $Z = a_1x_1 + a_2 = 3x_1 + 2x_2$ for a few values of Z. In this case, we take $Z = 0, Z = 60$ and $Z = 180$. These lines are called isoprofit lines. Once we have drawn the isoprofit line we can generate other lines by moving parallel to this line in the direction in which Z increases (for a max problem). The last isoprofit line intersecting (touching) the feasible region defines the largest Z-value and determines the optimal solution to the model.

The optimal(maximizing) solution is $x_1 = 20$, $x_2 = 60$, $Z = 180$. This problem has exactly one optimal solution.

Example 2. Solve the following linear programming problem:

$$\begin{aligned}
&\max z = 2x_1 + 2x_2 \\
&2x_1 + x_2 \le 100 \\
&x_1 + x_2 \le 80 \\
&x_1 \le 40 \\
&x_1 \ge 0 \\
&x_2 \ge 0
\end{aligned}$$
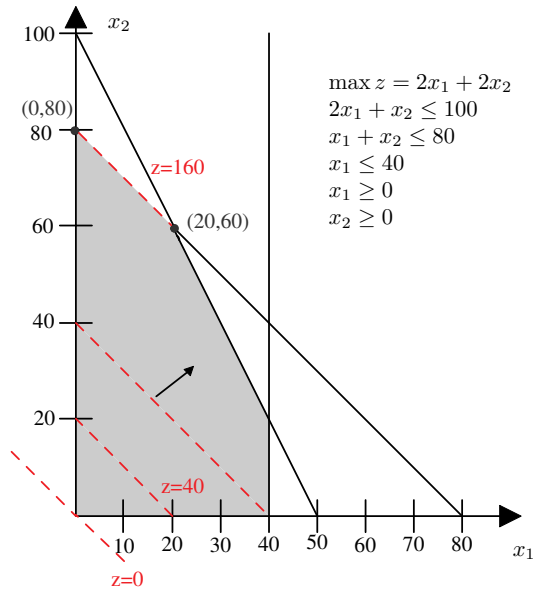
This problem has multiple optimal(or alternative optimal) solutions. This means that there is an infinite number of optimal solutions. (Every point on the line segment connecting the points (0,80) and (20,60) is optimal, with $Z = 160$). We may express the set of optimal solution as follows:

$$\begin{aligned}
x_1 &= 0t + 20(1 - t) \\
x_2 &= 80t + 60(1 - t) \\
t &\in [0, 1]
\end{aligned}$$

Example 3. Solve the following problem :

$$\begin{aligned}
&\max z = 2x_1 + 2x_2 \\
&2x_1 - x_2 \le 40 \\
&x_2 \ge 20 \\
&x_1 \ge 0 \\
&x_2 \ge 0
\end{aligned}$$
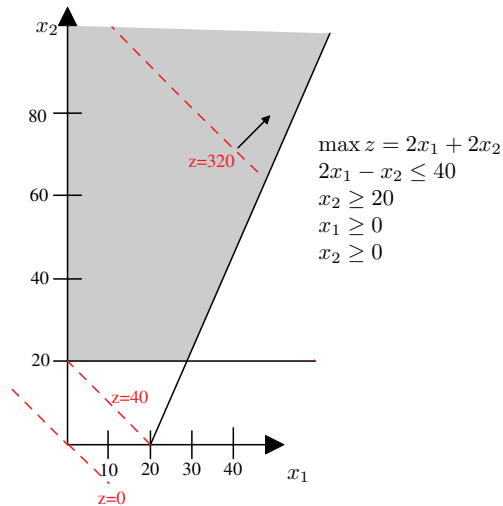
Unbounded Optimal Solution. The following problem has no optimal solution, because it is possible to find points in the feasible region with an arbitrarily large Z-value.

Example4. Solve the following problem:

$$\max z = 3x_1 + 2x_2$$
$$60x_1 + 40x_2 \leq 240$$
$$x_1 \geq 30$$
$$x_2 \geq 20$$
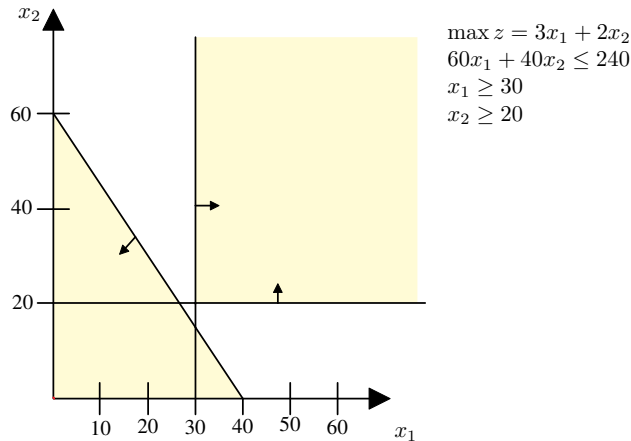
This problem has no optimal solution - the set of feasible solutions is empty (Empty Feasible Region). The system of inequalities(and/or equalities) defining the feasible region is inconsistent.

Observations From the graphical solution of two-variable linear programming problems we see that:

- The set of feasible solutions (feasible region)of a linear programming problem(LPP) is a **convex set**. The set $\mathbb{D}$ in n-dimensional space is a **convex set** if the line segment joining any pair of points in $\mathbb{D}$ is wholly contained in $\mathbb{D}$.

- There is only a finite number of extreme points(corner points) in the set of feasible solutions. For any convex set $\mathbb{D}$, a point P is an extreme point if each line segment that lies completely in $\mathbb{D}$ and contains the point P has P as an end point of the line segment.

- The optimal solution is attained (if it exists) in at least one extreme point.

Every LPP must fall into one of the following four cases:

1. There is only one optimal solution which is an extreme point (corner point) of the set of feasible solutions.

2. There are alternative optimal solutions (an infinite number of optimal solutions), but at least one optimal solution occurs at an extreme point (corner point) of the set of feasible solutions to the LPP.

3. The LPP is unbounded: There are points in the set of feasible solutions with an arbitrarily large Z- value (max problem) or arbitrarily small Z- value (min problem).

4. The LPP is infeasible: The set of feasible solutions is the empty set(The feasible region contains no points).

All the above facts can be proved for any LPP.

Chapter 3

Solving Linear Programming Problems: The Simplex Method

3.1 PREVIEW OF THE SIMPLEX METHOD

Before the simplex algorithm can be used to solve any linear programming problem (LPP), it must be converted into an equivalent **standard form** of LPP in which all the constraints are equations and all the following variables are non-negative.:

$$\max(\min)z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$
$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$$
$$\cdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m$$
$$x_i \geq 0, \ i = 1, \ldots, n$$

The system equations(constraints) can be written much more concisely in matrix form as follows: $\boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq 0$. We assume, that the rank of matrix $\boldsymbol{A}$ equals m. Any LPP can be converted into this standard form as follows:

1. We convert a ($\leq$) constraint $a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq b_i$ into an equality constraint by adding a **slack variable** s_i to the left-hand side of the constraint and adding the sign restriction $s_i \geq 0$: $a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n + s_i = b_i$, $s_i \geq 0$.

2. We convert ($\geq$) constraint $a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \geq b_i$ into an equality constraint by subtracting a **surplus variable** s_i from the left-hand side of the constraint and adding the sign restriction $s_i \geq 0$: $a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n - s_i = b_i$, $s_i \geq 0$.

3. (Dealing with unrestricted variables) If variable x_i can take both negative and nonnegative values, we use the substitution: $x_i = u_i - v_i$ and add two non-negativity constraints $u_i \geq 0$, $v_i \geq 0$.

Example. Convert the following linear programming problem into the standard form:

$$\max z = 2x_1 + 3x_2 - x_3$$
$$x_1 - 2x_2 \leq 5$$
$$x_2 + 3x_3 \geq 3$$
$$x_1 + x_2 - 2x_3 = 20$$
$$x_1, x_2 \geq 0$$

After converting constraints 1 and 2, we obtain:

$$\max z = 2x_1 + 3x_2 - x_3$$
$$x_1 - 2x_2 + s_1 = 5$$
$$x_2 + 3x_3 - s_2 = 3$$
$$x_1 + x_2 - 2x_3 = 20$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Next, we make the substitution $x_3 = u_3 - v_3$ and obtain the standard form of the problem:

$$\max z = 2x_1 + 3x_2 - u_3 + v_3$$
$$x_1 - 2x_2 + s_1 = 5$$
$$x_2 + 3u_3 + 3v_3 - s_2 = 3$$
$$x_1 + x_2 - 2u_3 + 2v_3 = 20$$
$$x_1, x_2, s_1, s_2, u_3, v_3 \geq 0$$

Converting from the graphical to the algebraic method

Graphical Method:

1. Graph all the constraints, including the nonnegativity restrictions - Suppose the solution space consists of infinite number of **feasible solutions**.

2. Identify **feasible corner points** (extreme points) of the solution space - Candidates for the optimal solution are given by a finite number of corner points.

3. Use the objective function to determine **the optimal corner point** from among all the candidates.

Algebraic Method:

1. Represent the solution space by m equations in n variables and restrict all the variables to nonnegative values, suppose $m \leq n$ and the system has infinite number of **feasible solutions**.

2. Determine a **feasible basic solution** of the equations - Candidates for the optimal solution are given by a finite number of **basic feasible solutions**.

3. Use the objective function to determine the **optimum basic feasible solution** from among all the candidates.

Definition 1. *Consider the set of equations $\boldsymbol{Ax} = \boldsymbol{b}$, if we set $n - m$ variables to zero and then solve the m equations for the remaining m variables, the resulting solution, if unique, is called a **basic solution** and must correspond to a (feasible or infeasible) corner point (extreme point) of the solution space. The $n - m$ variables set to zero are known as **nonbasic variables** (NBV) and the remaining m variables are called basic variables(BV).*

Remark The set of columns of the matrix $\boldsymbol{A}$ that corresponds to the basic variables is a linearly independent set of vectors. This set is called a basis and will

be denoted by $\mathbb{B}$.

Example. Find a few basic solutions to the following system of two equations with four variables:

$$x_1 + x_2 + 2x_4 = 3$$
$$2x_1 - x_2 - x_3 + 4x_4 = 1$$

We begin by choosing $BV = \{x_1, x_2\}$ to be basic variables and $NBV = \{x_3, x_4\}$ to be nonbasic variables. Setting $x_3 = x_4 = 0$, we obtain the following linear system:

$$x_1 + x_2 = 3$$
$$2x_1 - x_2 = 1$$

These equations lead to the unique (feasible) basic solution: $x_1 = \frac{4}{3}$, $x_2 = 1\frac{2}{3}$, $x_3 = 0$, $x_4 = 0$.
Next we choose $BV = \{x_2, x_3\}$ as our basic variables and set the nonbasic variables $NBV = \{x_1, x_4\}$ equal to zero, i.e. $x_1 = x_4 = 0$. We have the following system:

$$x_2 = 3$$
$$-x_2 - x_3 = 1$$

Solving the above system we obtain the (unfeasible) basic solution : $x_1 = 0$, $x_2 = 3$, $x_3 = -4$, $x_4 = 0$. If we choose $BV = \{x_1, x_4\}$ and $NBV = \{x_2, x_3\}$, we obtain the following system of linear equations:

$$x_1 + 2x_4 = 3$$
$$2x_1 + 4x_4 = 1$$

This system is inconsistent so the variables x_1, x_4 cannot simultaneously be basic variables!

Definition 2. *For any linear programming problem with m constraints, two basic feasible solutions are called **adjacent** if their sets of basic variables have $m-1$ basic variables in common.*

For example, the following two basic solutions from the above example $X_1 = (x_1 = \frac{4}{3}, x_2 = \frac{5}{3}, x_3 = 0, x_4 = 0)$ (feasible basic solution - FBS) and $X_2 = (x_1 = 0, x_2 = 3, x_3 = -4, x_4 = 0)$ (infeasible basic solution - IBS) are adjacent.

Definition 3. *Any basic solution to $\boldsymbol{Ax} = \boldsymbol{b}$ in which all the variables are non-negative is a **feasible basic solution**(FBS).*

Property 1. *For any LPP there is a unique extreme point(corner point) of the feasible region corresponding to each basic feasible solution. Also, there is at least one FBS corresponding to each extreme point of the feasible region.*

Theorem 1. *The feasible region of any LPP is a convex set. Also, if the LPP has an optimal solution, there must be an extreme point of the feasible region that is optimal.*

From the above facts, we see that in searching for an optimal solution to an LPP we need only find the best basic feasible solution(largest Z - value in a max problem or smallest Z - value in a min problem) to $\boldsymbol{Ax} = \boldsymbol{b}$.

3.2   THE IDEA OF THE SIMPLEX ALGORITHM

Step 1-Initialization: Find an initial Feasible Basic Solution(FBS) to the problem (this solution will be called the current FBS).

Step 2 - Optimality test: Is the current basic FBS optimal? If no, then go to the Iteration. If yes, then stop.

Step 3 Iteration: Perform an iteration to find an adjacent FBS to the current one that has a larger (not smaller) Z - value (for a max problem) and smaller Z value (for a min problem). Return to Step 2 using the new FBS as the current FBS.

There are three elements that must be determined using the simplex algorithm:

1. Finding an initial FBS.

2. Cheking if the current FBS is optimal? (Optimality test).

3. Transition from the current basic feasible solution to a better adjacent FBS.

Example Solve the following problem:

$$\max z = 4x_1 + 3x_2$$
$$x_1 + x_2 \leq 40$$
$$2x_1 + x_2 \leq 60$$
$$x_1, x_2 \geq 0$$
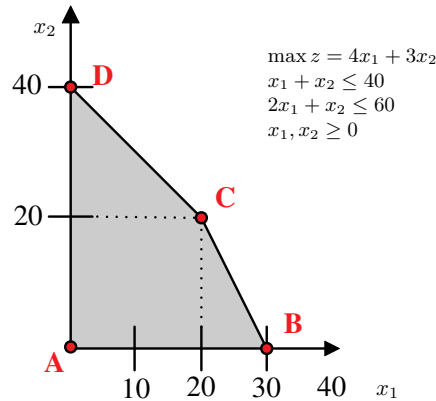
We convert the model into standard form by adding two slack variables s_1, s_2:

$$\max z = 4x_1 + 3x_2$$
$$s_1 \quad +x_1 \quad +x_2 \quad = 40$$
$$s_2 \quad +2x_1 \quad +x_2 \quad = 60$$
$$x_1, x_2, s_1, s_2 \geq 0$$

The above model is said to be in *basic (canonical) form*. We can obtain the first FBS by setting

$$BV = \{s_1, s_2\}, NBV = \{x_1, x_2\}$$

The initial feasible basic solution is $X^1 = (s_1 = 40, s_2 = 60, x_1 = 0, x_2 = 0)$ with value $Z = 0$. This FBS corresponds to the corner point **A** of the feasible region ABCD. We express this model in equation form as follows:

$$
\begin{array}{llll}
Z & -4x_1 & -3x_2 & = 0 \\
s_1 & +x_1 & +x_2 & = 40 \\
s_2 & +2x_1 & +x_2 & = 60 \\
x_1, x_2, s_1, s_2 \geq 0
\end{array}
\tag{3.1}
$$

Row 0 of the above system is the equation corresponding to the objective function $Z - 4x_1 - 3x_2 = 0$. There are no basic variables in the objective function.
This system of equations is represented by the **simplex tableau** as follows:

c_B	BV	s_1	s_2	x_1	x_2	$Solution$
0	s_1	1	0	1	1	40
0	s_2	0	1	2	1	60
	Z	0	0	-4	-3	0

The last row of the simplex tableau contains the so called optimality coefficients(except for the first and last columns) of the variables. We choose the nonbasic variable x_1 with the most negative coefficient in row Z (ties may be broken in an arbitrary fashion)- this variable is called *the entering variable*. We want to introduce this variable as a basic variable in the new adjacent FBS. Since each unit by which we increase x_1 increases the value of the objective function Z by 4,(for x_1 only by 3), we would like to make x_1 as large as possible. We see, from the second equation of (1), that x_1 cannot be greater than $x_1 = 40/1 = 40$ and from the third equation of (1) we get that x_1 cannot be greater than $x_1 = 60/2 = 30$. We choose minimum of these values ($x_1 = 30$). This means that basic variable s_2 will be a nonbasic variable in the new adjacent FBS. This variable is called *the leaving basic variable*. The new basic variables are $BV = \{s_1, x_1\}$. In the simplex tableau we add a new column at the end and element 2 in column x_1(the entering variable) and row s_2 (the leaving variable) is highlighted.

c_B	BV	s_1	s_2	$\vec{x_1}$	x_2		
0	s_1	1	0	1	1	40	$40/1 = 40$
0	$\vec{s_2}$	0	1	$\boxed{2}$	1	60	$60/2 = \underline{30}$
	Z	0	0	$\underline{-4}$	-3	0	

Now we solve the system of equations (3.1) with basic variables s_1, x_1. The process is based on Gauss-Jordan row operations. The column of the entering variable is known as the *pivot column* and the row of the leaving variable as the *pivot row*. The intersection of the pivot column and pivot row is the *pivot element* (element $\boxed{2}$). The Gauss-Jordan computations needed to produce the new basic solution include two types:

1. Pivot row

   a. Replace the leaving variable in the BV column with the entering variable.

   b. New pivot row = Current pivot row ÷ Pivot element

2. All the other rows, including Z:
   New row = (Current row) - (Its pivot column coefficient)× (New pivot row)

The new (second) simplex tableau is:

c_B	BV	s_1	s_2	x_1	x_2	
0	s_1	1	-0.5	0	0.5	10
4	x_1	0	0.5	1	0.5	30
	z	0	2	0	-1	120

The corresponding basic feasible solution is: $X^2 = (s_1 = 10, \ x_1 = 30, \ s_2 = 0, \ x_2 = 0)$ with the value of objective function $Z = 120$. This solution corresponds to the extreme (corner point) **B** of the feasible region. The solution X^2 is not optimal - the coefficient of the variable x_2 in the Z- row is negative (so x_2 will be the entering variable in the next iteration).

c_B	BV	s_1	s_2	x_1	$\vec{x_2}$		
0	$\vec{s_1}$	1	−0.5	0	$\boxed{0.5}$	10	$10/0.5 = \underline{20}$
4	x_1	0	0.5	1	0.5	30	$30/0.5 = 60$
	z	0	2	0	$\underline{-1}$	120	

So we have obtained the entering variable x_2 and the leaving variable s_1. The new basic variables are $BV = \{x_2, x_1\}$. We use the Gauss-Jordan row operations with the pivot element $\boxed{0.5}$. These computations produce the following simplex tableau:

c_B	BV	s_1	s_2	x_1	x_2	
3	x_2	2	−1	0	1	20
4	x_1	−1	1	1	0	20
	z	2	1	0	0	140

Now we have the corresponding FBS $X^3 = (x_2 = 20,\ x_1 = 20,\ s_1 = 0,\ s_2 = 0)$ with $Z = 140$. We find that this solution is optimal, because none of the coefficients in row-Z is negative, so the algorithm is finished. The feasible basic solution X^3 corresponds to the extreme point **C**.

3.3   UNBOUNDED SOLUTION

In some linear programming problems, the values of some variables may be increased indefinitely without violating any of the constraints - meaning that the feasible region is unbounded. As a result, the Z value may increase (maximization case) or decrease (minimization case) indefinitely. The following example shows how unboundedness can be recognized in the simplex tableau.
Example

$$\max z = 2x_1 + x_2 + x_3$$
$$s_1 \quad +3x_1 - x_2 = 60$$
$$s_2 \quad +x_1 - 2x_2 + 2x_3 = 10$$
$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

The first simplex tableau (BV=$\{s_1, s_2\}$)is as follows:

c_B	BV	s_1	s_2	x_1	x_2	x_3	FBS
0	s_1	1	0	3	−1	0	60
0	s_2	0	1	1	−2	2	10
	Z	0	0	−2	$\underline{-1}$	−1	0

We see that both x_1 and x_2 have negative Z- row coefficients. Hence increasing either one will improve the solution. If we take x_2 as the entering variable, we see that all the constraint coefficients in the x_2 column are negative or zero. This means that there is no leaving variable and that x_2 can be increased indefinitely without violating any of the constraints. Because each unit increase in x_2 will increase Z by 1, an infinite increase in x_2 leads to an infinite increase in Z.

Remark If there exists a variable with a negative coefficient in the Z- row of the simplex tableau and all the constraint coefficients in this variable's column are negative or zero then the objective function of the LPP is unbounded.

3.4 THE SIMPLEX ALGORITHM

1. The LPP must be first expressed in basic (canonical) form:

$$\max(\min)Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$
$$x_1 \qquad\qquad\qquad + a_{1m+1} x_{m+1} + \cdots + a_{1n} x_n = b_1$$
$$\qquad x_2 \qquad\qquad + a_{2m+1} x_{m+1} + \cdots + a_{2n} x_n = b_2$$
$$\cdots$$
$$\qquad\qquad x_m \ + a_{2m+1} x_{m+1} + \cdots + a_{2n} x_n = b_m$$
$$x_i \geq 0, \ i = 1, \ldots, n,$$

where the variables $x_1, \ldots, x_m$ are basic variables ($BV = \{x_1, \ldots, x_m\}$), the variables $x_{m+1}, x_{m+2}, \ldots, x_n$ are nonbasic variable ($NBV = \{x_{m+1}, x_{m+2}, \ldots, x_n\}$,) and $b_i \geq 0, \ i = 1, \ldots m (m < n)$.

2. The starting simplex tableau is:

c_B	BV	x_1	x_2	$\ldots$	x_m	x_{m+1}	$\ldots$	x_n	$Solution$
c_1	x_1	1	0	$\ldots$	0	a_{1m+1}	$\ldots$	a_{1n}	b_1
c_2	x_2	0	1	$\ldots$	0	a_{2m+1}	$\ldots$	a_{2n}	b_2
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
c_m	x_m	0	0	$\ldots$	1	a_{mm+1}	$\ldots$	a_{mn}	b_m
	Z	0	0	$\ldots$	0	$-\overline{c}_{m+1}$	$\ldots$	$-\overline{c}_n$	b_0

,

where:

$$\overline{c}_k = \sum_{i=1}^{m} c_i a_{ik} - c_k, \ k = 1, \ldots, n \qquad (3.2)$$

$$b_0 = \sum_{i=1}^{m} c_i b_i \qquad (3.3)$$

The numbers $\overline{c}_i$ are called *the optimality coefficients.*

3. If all the optimality coefficients $\overline{c}_1, \ldots, \overline{c}_n$ (all the Z- row coefficients) are nonnegative then STOP - the current basic solution is an optimal solution. Else, go to step 4.

4. If there exists a nonbasic variable with a negative optimality coefficient and all the coefficients of the column of the simplex tableau for this variable are nonpositive, then STOP - the objective function is unbounded. Else, go to step 5.

5. Select the entering variable x_p. In a maximization (minimization) problem this is the nonbasic variable having the most negative(positive) optimality coefficient (coefficient in the Z - row). Ties in this minimum(maximum) operation are broken arbitrarily. Select the leaving variable x_r. This is the basic variable x_r for which the following condition is satisfied:

$$\frac{b_r}{a_{rp}} = \min_{a_{ip}>0} \left\{ \frac{b_i}{a_{ip}} \right\}$$

The variable x_r will be a nonbasic variable in the next FBS .

6. Apply the Gauss-Jordan row operations with pivot element a_{rp}. Go back to step 3.

Example. Solve the following problem using the simplex algorithm:

$$\max z = 2x_1 + x_2 + x_3$$
$$3x_1 + x_2 + x_3 \leq 60$$
$$x_1 - x_2 + 2x_3 \leq 10$$
$$x_1 + x_2 - x_3 \leq 20$$
$$x_1, x_2, x_3 \geq 0$$

First we convert the problem into standard form and canonical form:

$$\max z = 2x_1 + x_2 + x_3$$
$$s_1 \qquad +3x_1 + x_2 + x_3 = 60$$
$$s_2 \qquad +x_1 - x_2 + 2x_3 = 10$$
$$s_3 \quad +x_1 + x_2 - x_3 = 20$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Next, we construct the first simplex tableau:

c_B	BV	s_1	s_2	s_3	x_1	x_2	x_3	*Solution*
0	s_1	1	0	0	3	1	1	60
0	s_2	0	1	0	1	-1	2	10
0	s_3	0	0	1	1	1	-1	20
	Z	0	0	0	-2	-1	-1	0

Remark It is easy to compute the optimality coefficients using formula(3.2). For example, for the variable x_1 we obtain : $\bar{c}_1 = 0*3 + 0*1 + 0*1 - 2 = -2$. If there are no basic variables in the objective function, then we have $\bar{c}_i = -c_i$, $i = 1, \ldots, n$ in the first simplex tableau.

c_B	BV	s_1	s_2	s_3	$\vec{x_1}$	x_2	x_3		
0	s_1	1	0	0	3	1	1	60	$60/3 = 20$
0	$\vec{s_2}$	0	1	0	$\boxed{1}$	-1	2	10	$10/1 = \underline{10}$
0	s_3	0	0	1	1	1	-1	20	$20/1 = 20$
	Z	0	0	0	$\underline{-2}$	-1	-1	0	

The entering variable and leaving variable are x_1 and s_2 respectively, so taking the pivot element to be $\boxed{1}$, the new simplex tableau can be computed by using the Gauss-Jordan row operations:

c_B	BV	s_1	s_2	s_3	x_1	x_2	x_3	*Solution*
0	s_1	1	-3	0	0	4	-5	30
2	x_1	0	1	0	1	-1	2	10
0	s_3	0	-1	1	0	2	-3	10
	Z	0	2	0	0	-3	3	20

It is easy to compute the Z - row coefficients using formula (3.2). For example, the x_2 optimality coefficient is computed as follows: $0*4 + 2*(-1) + 0*2 - 1 = -3$. The value of the objective function (b_0) is: $0*30 + 2*10 + 0*10 = 20$. The next iterations are given by the following simplex tableaux:

c_B	BV	s_1	s_2	s_3	x_1	$\vec{x_2}$	x_3		
0	s_1	1	-3	0	0	4	-5	30	$30/4 = 7.5$
2	x_1	0	1	0	1	-1	2	10	$-$
0	$\vec{s_3}$	0	-1	1	0	$\boxed{2}$	-3	10	$10/2 = \underline{5}$
	Z	0	2	0	0	$\underline{-3}$	3	20	

c_B	BV	s_1	s_2	s_3	x_1	x_2	$\vec{x_3}$		
0	$\vec{s_1}$	1	-1	-2	0	0	$\boxed{1}$	10	$10/1 = \underline{10}$
2	x_1	0	0.5	0.5	1	0	0.5	15	$15/0.5 = 30$
1	x_2	0	-0.5	0.5	0	1	-1.5	5	$-$
	Z	0	-0.5	0.5	0	0	$\underline{-1.5}$	35	

c_B	BV	s_1	$\vec{s_2}$	s_3	x_1	x_2	x_3		
1	x_3	1	-1	-2	0	0	1	10	$-$
2	$\vec{x_1}$	-0.5	$\boxed{1}$	1.5	1	0	0	10	$10/1 = \underline{10}$
1	x_2	1.5	-2	-2.5	0	1	0	20	$-$
	Z	1.5	$\underline{-1}$	-1.5	0	0	0	50	

c_B	BV	s_1	$\vec{s_2}$	s_3	x_1	x_2	x_3	$Solution$
1	x_3	0.5	0	-0.5	1	0	1	20
0	s_2	-0.5	1	1.5	1	0	0	10
1	x_2	0.5	0	0.5	2	1	0	40
	Z	1	0	0	1	0	0	60

The last simplex tableau contains the optimal solution: $x_3 = 20$, $s_2 = 10$, $x_2 = 40$, $x_1 = 0$, $s_1 = 0$, $s_2 = 0$ with the optimal value of objective function $Z = 60$. This tableau is called the optimal tableau.

3.5 MULTIPLE OPTIMAL SOLUTIONS

If there is a nonbasic variable which has an optimality coefficient equal to zero in the optimal simplex tableau then multiple optima exist.

Example.

$$\max Z = 2x_1 + 2x_2$$
$$2x_1 + x_2 \le 100$$
$$x_1 + x_2 \le 80$$
$$x_1 \le 40$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$
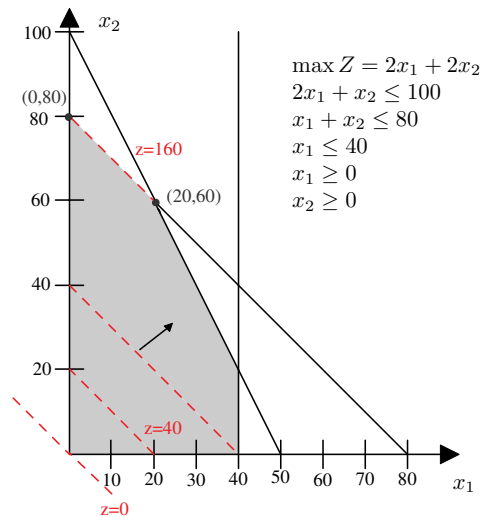
One optimal simplex tableau is given below:

c_B	BV	s_1	s_2	s_3	$\vec{x_1}$	x_2		
0	$\vec{s_1}$	1	-1	0	$\boxed{1}$	0	20	$20/1 = \underline{20}$
2	x_2	0	1	0	1	1	80	$80/1 = 80$
0	s_3	0	0	1	1	0	40	$40/1 = 40$
	Z	0	2	0	$\underline{0}$	0	160	

The associated optimal basic solution is $X^{opt} = (x_1, x_2, s_1, s_2, s_3) = (0, 80, 20, 0, 40)$ with $Z = 169$. This optimal FBS corresponds to the extreme (corner point) point $X^1 = (0, 80)$ on the picture. The nonbasic variable x_1 has a zero coefficient in the Z-row of this optimal simplex tableau. Taking the variable x_1 as the entering variable and carrying out a simplex iteration (with leaving variable s_1), we obtain the following optimal simplex tableau:

c_B	BV	s_1	s_2	s_3	x_1	x_2	$Solution$
2	x_1	1	-1	0	1	0	20
2	x_2	-1	2	0	0	1	60
0	s_3	-1	1	1	0	0	20
	Z	0	2	0	0	0	160

This iteration gives us a new optimal feasible basic solution $x_1 = 20$, $x_2 = 60$, $s_3 = 20$, $s_1 = s_2 = 0$ and $Z = 160$ which corresponds to the point $X^2 = (20, 60)$. The simplex method determines only two optimal basic solutions (corner points). Each optimal solution $X = (x_1, x_2)$ can be determined as a linear combination of points X^1 and X^2: $X = t * X^1 + (1 - t) * X^2$.
$x_1 = 0t + 20(1 - t)$,
$x_2 = 80t + 60(1 - t)$, $t \in [0, 1]$.

3.6 USING THE SIMPLEX ALGORITHM TO SOLVE MINIMIZATION PROBLEMS

Simple modifications are needed to solve a minimization problem: If all the non-basic variables in the Z- row of the simplex tableau have nonpositive coefficients, then the current FBS is optimal. If any nonbasic variable in the Z - row has the positive coefficient, choose the variable with the *most positive* coefficient in the Z - row of the simplex tableau as the entering variable. Another way is to multiply the objective function equation by (-1) and solve the resulting maximization problem

Example. We illustrate the method by solving the following linear problem:

$$\min Z = -3x_1 + x_2$$
$$3x_1 + x_2 \leq 6$$
$$-x_1 + 2x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

We convert the problem to standard and canonical form:

$$\min Z = -3x_1 + x_2$$
$$s_1 \quad +3x_1 + x_2 = 6$$
$$s_2 \quad -x_1 + 2x_2 = 1$$
$$x_1, x_2 \geq 0$$

c_B	BV	s_1	s_2	$\vec{x_1}$	x_2		
0	$\vec{s_1}$	1	0	$\boxed{3}$	1	6	$6/3 = 2$
0	s_2	0	1	-1	2	1	$-$
	Z	0	0	$\underline{3}$	-1	0	

c_B	BV	s_1	s_2	$\vec{x_1}$	x_2	
3	x_1	$1/3$	0	1	$1/3$	2
0	s_2	$1/3$	1	0	$7/3$	3
	Z	-1	0	0	-2	-6

The last simplex tableau is optimal: all coefficients of the Z row are nonpositive. The optimal solution is: $x_1 = 2$, $x_2 = 0$, $s_1 = 0$, $s_2 = 3$. The minimum value of the objective function value is $Z = -6$.

Chapter 4

Artificial Starting Solutions

A linear programming problem in which all the constraints are ($\leq$) with nonnegative right-hand sides gives an initial "all slack" starting basic feasible solution with all the slack variables being basic. Problems with ($=$) and/or ($\geq$) constraints need to use artificial variables at the beginning of the simplex algorithm. There are two methods: the M-method(also called the Big M-method) and the Two-Phase method.

4.1   THE BIG M METHOD

Example. Solve the problem:

$$\max Z = 2x_1 + x_2 - 3x_3$$
$$x_1 + x_2 + x_3 \geq 6$$
$$2x_1 + x_2 = 14$$
$$x_1, x_2, x_3 \geq 0$$

After converting to standard form, we have:

$$\max Z = 2x_1 + x_2 - 3x_3$$
$$x_1 + x_2 + x_3 - s_1 = 6$$
$$2x_1 + x_2 = 14$$
$$x_1, x_2, x_3, s_1 \geq 0$$

The above system of equations is not in basic form - not all the equations contain a basic variable. If equation i does not have a slack variable (or a variable that can play such a role), an artificial variable, a_i, is added to form a starting solution similar to the all-slack basic solution. In the example considered, we add two artificial variables a_1 and a_2. However, because the artificial variables are not part of the original linear model, they are assigned a very high **penalty** in the objective function, thus forcing them (eventually) to equal zero in the optimal solution.
Penalty Rule for Artificial Variables.
Given M, a sufficiently large positive value (mathematically, $M \to \infty$), the objective coefficient of an artificial variable represents an appropriate penalty if:

Coefficient of an artificial variable in the objective function = $\begin{cases} -M, & \text{in maximization problems} \\ M, & \text{in minimization problems.} \end{cases}$

So we have:

$$\max Z = 2x_1 + x_2 - 3x_3 - Ma_1 - Ma_2$$
$$a_1 \quad +x_1 + x_2 + x_3 - s_1 = 6$$
$$a_2 \quad +2x_1 + x_2 = 14$$
$$x_1, x_2, x_3, s_1, a_1, a_2 \geq 0$$

We construct the first simplex tableau using formula (3.2)).

c_B	BV	a_1	a_2	$\vec{x_1}$	x_2	x_3	s_1	Solution	
$-M$	$\vec{a_1}$	1	0	$\boxed{1}$	1	1	-1	6	$6/1 = \underline{6}$
$-M$	a_2	0	1	2	1	0	0	14	$14/2 = 7$
	Z	0	0	$\underline{-3M-2}$	$-2M-1$	$-M+3$	M	$-20M$	

Continuing with the simplex computations, we will obtain the following simplex tableaux:

c_B	BV	a_1	a_2	x_1	x_2	x_3	$\vec{s_1}$	Solution	
2	x_1	1	0	1	1	1	-1	6	$-$
$-M$	$\vec{a_2}$	-2	1	0	-1	-2	$\boxed{2}$	2	$2/2 = \underline{1}$
	Z	$3M+2$	0	0	$M+1$	$2M-1$	$\underline{-2M-2}$	$-2M+12$	

c_B	BV	a_1	a_2	x_1	x_2	x_3	s_1	Solution
2	x_1	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	7
0	s_1	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	1	1
	Z	M	$M+1$	0	0	3	0	14

The last tableau is optimal. The optimal solution is: $x_1 = 7$, $s_1 = 1$, $x_2 = x_3 = 0$ with $Z = 14$.

Remark The use of the penalty M will not force an artificial variable to equal zero in the final simplex iteration if the LPP does not have a feasible solution (i.e. the constraints are not consistent). In this case, the final simplex tableau will include at least one artificial variable with a positive value.

Example. Solve the problem :

$$\max Z = 2x_1 + 2x_2$$
$$6x_1 + 4x_2 \leq 24$$
$$x_1 \geq 5$$
$$x_1, x_2 \geq 0$$

The standard form is:
$$\max Z = 2x_1 + 2x_2$$
$$s_1 \quad +6x_1 + 4x_2 = 24$$
$$x_1 - s_2 = 5$$
$$x_1, x_2, s_1, s_2 \geq 0$$

We use the Big M method (it is enough to add only one artificial variable):

$$\max Z = 2x_1 + 2x_2 - Ma_2$$
$$s_1 \quad +6x_1 + 4x_2 = 24$$
$$a_2 \quad +x_1 - s_2 = 5$$
$$x_1, x_2, s_1, s_2 \geq 0$$

c_B	BV	s_1	a_2	$\vec{x_1}$	x_2	s_2	Solution	
0	$\vec{s_1}$	1	0	$\boxed{6}$	4	0	24	$24/6 = \underline{4}$
$-M$	a_2	0	1	1	0	-1	5	$5/1 = 5$
	Z	0	0	$\underline{-M-2}$	-2	M	$-5M$	

c_B	BV	s_1	a_2	x_1	x_2	s_2	Solution
2	x_1	$\frac{1}{6}$	0	1	$\frac{2}{3}$	0	4
$-M$	a_2	$-\frac{1}{6}$	1	0	$-\frac{2}{3}$	-1	1
	Z	$\frac{1}{6}M + \frac{1}{3}$	0	0	$\frac{2}{3}M - \frac{2}{3}$	M	$-M + 8$

The last simplex tableau is optimal - all the Z- row coefficients are nonnegative. The artificial variable a_2 is a basic variable with positive value, so the problem **is not consistent** - it has no feasible solutions.

4.2 THE TWO-PHASE METHOD

This method solves a linear programming problem in two phases: Phase I attempts to find a starting feasible solution and, if one is found, Phase II is used to solve the original problem.

Phase I Put the problem in equation form, and add the necessary artificial variables to the constraints to get a starting feasible basic solution. Next, find a basic solution of the resulting equations that, regardless of whether the problem is to minimize or maximize, **always** minimizes the sum of the artificial variables. If the minimum value of this sum is positive, the problem has no feasible solution, which ends the process. Otherwise, proceed to Phase II.

Phase II Use the feasible solution from Phase I as a starting feasible basic solution for the original problem.

We use this method to solve the following problem:

$$\max Z = 4x_1 + x_2$$
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Using s_2 as a surplus in the second constraint and s_3 as a slack variable in the third constraint, the standard form of the problem is:

$$\max Z = 4x_1 + x_2$$
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 - s_2 = 6$$
$$x_1 + 2x_2 + s_3 = 4$$
$$x_1, x_2, s_2, s_3 \geq 0$$

The third equation has a basic variable (s_3), but the first and second do not. Thus we add the artificial variables a_1 and a_2 and we minimize the sum of the artificial variables $a_1 + a_2$. The resulting model is:

$$\min Z^a = a_1 + a_2$$
$$3x_1 + x_2 \qquad +a_1 \qquad = \quad 3$$
$$4x_1 + 3x_2 \quad -s_2 \qquad +a_2 \qquad = \quad 6$$
$$x_1 + 2x_2 \qquad\qquad\qquad +s_3 \quad = \quad 4$$
$$x_1, x_2, s_2, s_3, a_1, a_2 \geq 0$$

Taking a_1, a_2, s_3 as basic variables and using formula (2) to compute the coefficients of the Z^a - row, we obtain the first simplex tableau:

c_B	BV	x_1	x_2	s_2	a_1	a_2	s_3	$Solution$
1	a_1	3	1	0	1	0	0	3
1	a_2	4	3	−1	0	1	0	6
0	s_3	1	2	0	0	0	1	4
	Z^a	7	4	−1	0	0	0	9

Now we use the simplex algorithm for a minimization problem and obtain (after two iterations) the following optimal tableau:

c_B	BV	x_1	x_2	s_2	a_1	a_2	s_3	$Solution$
0	x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
0	x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
0	s_3	0	0	1	1	−1	1	1
	Z^a	0	0	0	−1	−1	0	0

Because the minimum value of $Z^a = 0$, Phase I produces the basic feasible solution $x_1 = \frac{3}{5}, x_2 = \frac{6}{5}, s_3 = 1$. We can eliminate the columns for a_1 and a_2 from the tableau and move to Phase II.

Phase II After deleting the artificial columns, we write the original problem as

$$
\begin{aligned}
\max Z = 4x_1 + x_2 \\
x_1 \qquad + \tfrac{1}{5}s_2 \qquad\quad &= \tfrac{3}{5} \\
x_2 - \tfrac{3}{5}s_2 \qquad\quad &= \tfrac{6}{5} \\
s_2 + s_3 &= 1 \\
x_1, x_2, s_2, s_3 \geq 0
\end{aligned}
$$

Now we can begin the simplex algorithm with the following simplex tableau:

c_B	BV	x_1	x_2	s_2	s_3	$Solution$
4	x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
1	x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
0	s_2	0	0	1	1	1
	Z	0	0	$\frac{1}{5}$	0	$\frac{18}{5}$

Because we are maximizing, this is the optimal tableau. The optimal solution is $x_1 = \frac{3}{5}, x_2 = \frac{6}{5}$ and $Z = \frac{18}{5}$.

Remark All commercial packages use the two-phase method to solve linear programming problems.

The removal of the artificial variables and their columns at the end of Phase I can take place only when they are all nonbasic. If one or more artificial variables are basic (but equal to zero) at the end of Phase I, then the following additional steps must be undertaken to remove them prior the start of Phase II.

Step 1. Select an artificial variable equal to zero to leave the set of basic variables (choosing the leaving variable) and designate its row as the pivot row. The entering variable can be any nonbasic variable with a nonzero(positive or negative) coefficient in the pivot row. Perform the associated simplex iteration.

Step 2. Remove the column of the artificial variable that has just left from the tableau. If all the artificial variables equal to zero have been removed, go to Phase II. Otherwise, go back to Step 1.

4.3 SENSITIVITY ANALYSIS

Sensitivity analysis is concerned with how changes in the parameters of the linear programming model affect the optimal solution.

Example. Let us consider the following problem (described in basic form) of manufacturing two products W_1 and W_2 from two raw materials S_1, S_2:

$$
\begin{aligned}
\max Z = 3x_1 + 2x_2 \quad & \text{[Maximizing the revenue]} \\
s_1 \qquad +2x_1 + x_2 = 100 \quad & \text{[Limit on raw material } S_1] \\
s_2 \qquad +x_1 + x_2 = 80 \quad & \text{[Limit on raw material } S_2] \\
s_3 \ +x_1 = 40 \quad & \text{[Demand for } W_1] \\
x_1, x_2, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

The optimal solution of the problem is $x_1 = 20$, $x_2 = 60$ with $Z = 180$. We want to determine the range of values of a parameter (e.g. a coefficient in the objective function - the unit price of product W_1 in our example) over which the optimal solution remains unchanged. Solving this problem using the simplex algorithm, we obtain the following optimal tableau:

c_B	BV	s_1	s_2	s_3	x_1	x_2	Solution
3	x_1	**1**	**-1**	**0**	1	0	20
2	x_2	**-1**	**2**	**0**	0	1	60
0	s_3	**-1**	**1**	**1**	0	0	20
	Z	1	1	0	0	0	180

Let us assume that the unit price of $W1$ equals $3 + \delta$. Then we have:

c_B	BV	s_1	s_2	s_3	x_1	x_2	Solution
$3+\delta$	x_1	1	-1	0	1	0	20
2	x_2	-1	2	0	0	1	60
0	s_3	-1	1	1	0	0	20
	Z	$1+\delta$	$-\delta+1$	0	0	0	180

The optimal solution remains optimal as long as all the $Z-$ row coefficients are nonnegative, so we get:

$$
\begin{cases}
1 + \delta \geq 0 \\
-\delta + 1 \geq 0
\end{cases}
$$

Solving this system of inequalities we obtain $\delta \in [-1, 1]$. Thus the optimal solution will remain optimal when the price of W_1 belongs to the interval [2,4].

Now we examine how the optimal solution to the LPP changes if the right-hand side of a constraint is changed. For example, we consider the constraint for raw material S_1 (coefficient $b_1 = 100$). Changing b_i will leave the Z - row of the simplex tableau unchanged, but will change the the optimal solution. If at least one variable becomes negative, the current solution is no longer feasible. So we look for the values of δ for which the following system of equalities is consistent:

$$
\begin{cases}
2x_1 + x_2 = 100 + \delta \\
x_1 + x_2 = 80 \\
s_3 + x_1 = 40 \\
x_1, x_2, s_1, s_2, s_3 \geq 0
\end{cases}
$$

We describe this system in matrix form:

$$
\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 100 + \delta \\ 80 \\ 40 \end{bmatrix}
$$

Multiplying both sides of these equations by the inverse to the matrix of coefficients, we get:

$$\begin{bmatrix} x_1 \\ x_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 100 + \delta \\ 80 \\ 40 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The inverse matrix we can be obtained from the simplex tableau. It consists of the columns in the optimal tableau that correspond to the initial basic variables $BV = \{s_1, s_2, s_3\}$(taken in the same order):

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

Hence, we get:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 100 + \delta \\ 80 \\ 40 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For the current set of basic variables (basis) to remain optimal, we require that the following system of inequalities holds:

$$20 + \delta \geq 0$$
$$60 - \delta \geq 0$$
$$20 - \delta \geq 0$$

The set of basic variables at the optimal solution will be (x_1, x_2, s_3) as long as $\delta \in [-20, 20]$ i.e. if between 80 and 120 units of row material $S1$ ($b_1 \in [80, 120]$, where the current amount is $b_1 = 100$) are available.

Chapter 5

Dual Problem

5.1 FINDING THE DUAL OF A LINEAR PROGRAMMING PROBLEM

Associated with any linear programming problem, there is another linear problem, called the dual problem (Dual in short). Now we explain how to find the dual problem to a given LPP, we discuss the economic interpretation of the dual problem and the relation that exists between an LPP (called the Primal) and its dual problem. We consider the LPP in normal (canonical) form :

$$\max Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \leq & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \leq & b_2 \\
\vdots & & \vdots & & & & \vdots & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \leq & b_m
\end{array}
$$

$$x_j \geq 0 \ (j = 1, 2, \ldots, n)$$

The original problem is called the primal. The dual problem is defined as follows:

$$\min W = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

$$
\begin{array}{ccccccccc}
a_{11}y_1 & + & a_{21}y_2 & + & \cdots & + & a_{m1}y_m & \geq & c_1 \\
a_{12}y_1 & + & a_{22}y_2 & + & \cdots & + & a_{m2}y_m & \geq & c_2 \\
\vdots & & \vdots & & & & \vdots & & \\
a_{1n}y_1 & + & a_{2n}y_2 & + & \cdots & + & a_{mn}y_m & \geq & c_n
\end{array}
$$

$$y_i \geq 0 \ (i = 1, 2, \ldots, m)$$

We illustrate the construction of the dual problem from the primal problem in table 5.1

Example The Company FURNITURE manufactures tables and chairs. A table sells for \$160, and a chair sells for \$200. The demand for tables and chairs is unlimited. The manufacture of each type of furniture requires labor, lumber, and inventory space. The amount of each resource needed to make tables and chairs and the daily limits of available resources is given in the following table:

Resource	Resources needed		Amount of resource available(hours)
	Table	Chair	
Labor(hours)	2	4	40
Lumber($(feet)^3$)	18	18	216
Inventory space($(feet)^2$)	24	12	240

Table 5.1: Construction of the Dual from the Primal

$\min W$		$(x_1 \geq 0)$	$(x_2 \geq 0)$	$\cdots$	$(x_n \geq 0)$	
		x_1	x_2		x_n	
$(y_1 \geq 0)$	y_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$\leq b_1$
$(y_2 \geq 0)$	y_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$\leq b_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$(y_m \geq 0)$	y_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$\leq b_m$
		$\geq c_1$	$\geq c_2$		$\geq c_n$	

The table is headed by $\max Z$ spanning the x columns.

FURNITURE wants to maximize total revenue.
Primal problem:

$$\max z = 160x_1 + 200x_2$$
$$\begin{aligned} 2x_1 + 4x_2 &\leq 40 \\ 18x_1 + 18x_2 &\leq 216 \\ 24x_1 + 12x_2 &\leq 240 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

Dual problem

$$\min w = 40y_1 + 216y_2 + 240y_3$$
$$\begin{aligned} 2y_1 + 18y_2 + 24y_3 &\geq 160 \\ 4y_1 + 18y_2 + 12y_3 &\geq 200 \\ y_1,\ y_2,\ y_3 &\geq 0. \end{aligned}$$

5.1.1 ECONOMIC INTERPRETATION OF THE DUAL PROBLEM

Suppose there is an entrpreneur who wants to purchase all of FURNITURE's resourses i.e. 40 hours of labor, 210 $(feet)^3$ of lumber and $240(feet)^2$ of inventory space. Then he must determine the price he is willing to pay for a unit of each of FURNITURE's resources. Let y_1, y_2 and y_3 be the price for one hour of labor, one cubic feet of lumber and one square feet of inventory space. We show that the prices of resources should be determined by solving the dual problem. The total price the entrepreneur must pay for the resources is $40y_1 + 216y_2 + 240y_3$ and since he wishes to minimize the cost of purchasing the resources, he wants to:

$$\min W = 40y_1 + 216y_2 + 240y_3.$$

But he must be willing to pay enough to induce FURNITURE to sell its resources. For example, he must offer FURNITURE at least $160 for a combination of resources that includes 2 hours of labor, 18 cubic feet of lumber and 24 square feet of inventory space, because FURNITURE could use these resources to manufacture a table that can be sold for $160. Since he is offers $2y_1 + 18y_2 + 24y_3$ for the resources used to produce a table, he must choose y_1, y_2, y_3 to satisfy

$$2y_1 + 18y_2 + 24y_3 \geq 160.$$

Similar reasoning applied to the resources required for a chair gives:

$$4y_1 + 18y_2 + 12y_3 \geq 200.$$

The sign restrictions $y_1, y_2, y_3 \geq 0$ must also hold. So we see that the solution to the dual of Furnitures's problem yields prices for labor, lumber and inventory space.

Finding the dual problem to any LPP - An Example

$$
\begin{array}{rcl}
\max z = 2x_1 + x_2 & & \min w = 2y_1 + 3y_2 + y_3 \\
x_1 + x_2 = 2 & \longleftrightarrow & y_1 \text{ - Unrestricted} \\
2x_1 - x_2 \geq 3 & \longleftrightarrow & y_2 \leq 0 \\
x_1 - x_2 \leq 1 & \longleftrightarrow & y_3 \geq 0 \\
x_1 \geq 0 & \longleftrightarrow & y_1 + 2y_2 + y_3 \geq 2 \\
x_2 - \text{ Unrestricted} & \longleftrightarrow & y_1 - y_2 - y_3 = 1.
\end{array}
$$

The general conclusion from the example above is that the variables and constraints in the primal and dual problems are defined by the rules in the following table:

Maximization problem		Minimization problem
Constraints		*Variables*
$\geq$	$\leftrightarrow$	≤ 0
$\leq$	$\leftrightarrow$	≥ 0
$=$	$\leftrightarrow$	*Unrestricted*
Variables		*Constraints*
≥ 0	$\leftrightarrow$	$\geq$
≤ 0	$\leftrightarrow$	$\leq$
Unrestricted		$=$

Table 5.2: Rules for constructing the Dual Problem

5.1.2 PRIMAL-DUAL RELATIONSHIPS

Now the key relationships between the primal and dual problems will be described:

Theorem 2. *The dual of the dual problem yields the original primal problem.*

Theorem 3 (Weak duality property)**.** *If we choose any feasible solution to the primal and any feasible solution to the dual, the $W-$ value for the feasible dual solution will be at least as large as the $Z-$ value for the feasible primal solution. Let $\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^T$ be any feasible solution to the primal and $\boldsymbol{y} = [y_1, y_2, \ldots, y_m]$ be any feasible solution to the dual. Then*

$$(Z - value \text{ for } \boldsymbol{x}) \leq (W - value \text{ for } \boldsymbol{y})$$

Two properties result from this theorem:

Property 2. *If $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ and $\bar{\boldsymbol{y}} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)$ are feasible solutions of the primal problem and dual problem respectively such that $Z = c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_n\bar{x}_n = b_1\bar{y}_1 + b_2\bar{y}_2 + \ldots + b_m\bar{y}_m = W$, then $\bar{\boldsymbol{x}}$ must be an optimal solution to the primal problem and $\bar{\boldsymbol{y}}$ must be an optimal solution to the dual problem.*

Property 3. *If the primal(dual) is unbounded, then the dual(primal) problem is infeasible.*

Theorem 4 (Duality theorem)**.** *The following are the only possible relations between the primal and dual problems:*

1. *If one problem has feasible solutions and a bounded objective function (and so has the optimal solution), then so does the other problem.*

2. *If one problem has feasible solutions and an unbounded objective function(and so no optimal solution), then the other problem has no feasible solutions.*

3. If one problem has no feasible solutions, then the other problem has either no feasible solution or an unbounded objective function.

Now we can give an interpretation of the dual problem to the maximization problem. We know that for optimal solutions $\bar{\mathbf{x}}$ of the primal problem and $\bar{\mathbf{y}}$ of the dual problem the following equality holds:

$$Z = c_1\bar{x}_1 + c_2\bar{x}_2 + \ldots + c_n\bar{x}_n = b_1\bar{y}_1 + b_2\bar{y}_2 + \ldots + b_m\bar{y}_m = W$$

So each $b_i\bar{y}_i$ can be interpreted as the contribution to profit by having b_i units of resource i available in the primal problem. Thus, the optimal value of the dual variable $\bar{y}_i$ (it is called its shadow price) can be interpreted as the contribution to profit by a unit of resource $i(i = 1, 2, \ldots, m)$.

5.2   READING THE OPTIMAL SOLUTION TO THE DUAL FROM THE Z-ROW OF THE OPTIMAL SIMPLEX TABLEAU IF THE PRIMAL IS A MAX PROBLEM

After solving the primal problem using the simplex method, we can read the optimal dual solution from the optimal simplex tableau. Let us consider the LPP for the firm FURNITURE. The optimal simplex tableau is as follows:

Table 5.3: Optimal simplex tableau

		s_1	s_2	s_3	x_1	x_2	
200	x_2	$\frac{1}{2}$	$-\frac{1}{18}$	0	0	1	8
160	x_1	$-\frac{1}{2}$	$\frac{1}{9}$	0	1	0	4
0	s_3	6	-2	1	0	0	48
	Z	20	$\frac{20}{3}$	0	0	0	2240

The basic variables are $ZB = \{x_2, x_1, s_3\}$ and the basis is $\mathbb{B} = \begin{bmatrix} 4 & 2 & 0 \\ 18 & 18 & 0 \\ 12 & 24 & 1 \end{bmatrix}$.

The inverse matrix to $\mathbb{B}$ is: $\mathbb{B}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{18} & 0 \\ -\frac{1}{2} & \frac{1}{9} & 0 \\ 6 & -2 & 1 \end{bmatrix}$ and columns s_1, s_2, s_3 of the optimal simplex tableau contain this matrix. The optimal solution to the dual problem $(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ can be computed using the matrix $\mathbb{B}^{-1}$ as follows:

$$(\bar{y}_1, \bar{y}_2, \bar{y}_3) = c_B\mathbb{B}^{-1} = (200, 160, 0) \begin{bmatrix} \frac{1}{2} & -\frac{1}{18} & 0 \\ -\frac{1}{2} & \frac{1}{9} & 0 \\ 6 & -2 & 1 \end{bmatrix} = (20, \frac{20}{3}, 0).$$

where the vector c_B contains the coefficients of the objective function corresponding to the basic variables. The optimal to the dual is given by the coefficients of variables s_1, s_2, s_3 in the $Z-$ row in the optimal simplex tableau. If the primal problem is of any form , then the optimal solution to the dual may be read from the $Z-$ row of optimal simplex tableau by using the following rules:

- The optimal value of dual variable y_i if constraint i is of the type $\leq$ equals the coefficient of s_i in the $Z-$ row of the optimal simplex tableau.

- The optimal value of dual variable y_i if constraint i is of the type $\geq$ is the negative of coefficient of e_i in the $Z-$ row of the optimal simplex tableau), where e_i is a surplus variable.

- The optimal value of dual variable y_i if constraint i is an equality = (coefficient of a_i in the $Z-$ row of the optimal simplex tableau)-M, where a_i is an artificial variable.

Example.

$$\max z = 3x_1 + 2x_2 + 5x_5$$
$$\begin{aligned} x_1 + 3x_2 + 2x_3 &\leq 15 \\ 2x_2 - x_3 &\geq 5 \\ 2x_1 + x_2 - 5x_3 &= 10 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

To solve this problem we use the M method:

$$\max Z = 3x_1 + 2x_2 + 5x_5 - Ma_2 - Ma_3$$
$$\begin{aligned} x_1 + 3x_2 + 2x_3 + s_1 &= 15 \\ 2x_2 - x_3 - e_2 + a_2 &= 5 \\ 2x_1 + x_2 - 5x_3 + a_3 &= 10 \\ x_1, x_2, x_3, s_1, a_2, a_3 &\geq 0. \end{aligned}$$

After carrying out a few iterations we obtain the optimal simplex tableau:

Table 5.4: M-method, the last simplex tableau

		x_1	x_2	x_3	s_1	e_2	a_2	a_3	
5	x_3	0	0	1	$\frac{4}{23}$	$\frac{5}{23}$	$-\frac{5}{23}$	$-\frac{2}{23}$	$\frac{15}{23}$
2	x_2	0	1	0	$\frac{2}{23}$	$-\frac{9}{23}$	$\frac{9}{23}$	$-\frac{1}{23}$	$\frac{65}{23}$
3	x_1	1	0	0	$\frac{9}{23}$	$\frac{17}{23}$	$-\frac{17}{23}$	$\frac{7}{23}$	$\frac{120}{23}$
Z		0	0	0	$\frac{51}{23}$	$\frac{58}{23}$	$M - \frac{58}{23}$	$M + \frac{9}{23}$	$\frac{565}{23}$

The dual problem has the following form:

$$\min W = 15y_1 + 5y_2 + 10y_3$$
$$\begin{aligned} y_1 + 2y_3 &\geq 3 \\ 3y_1 + 2y_2 + y_3 &\geq 2 \\ 2y_1 - y_2 - 5y_3 &\geq 5 \end{aligned}$$
$$y_1 \geq 0, y_2 \leq 0, y_3 - \text{ Unrestricted.}$$

Reading the optimal solution to the dual from the optimal simplex tableau we get:

- The first constraint is a '$\leq$' inequality, so $y_1 = \frac{51}{23}$ (coefficient of s_1 in the $Z-$ row).

- The second constraint is a '$\geq$' inequality, so $y_2 = -\frac{58}{23}$ (- coefficient of e_2 in the $Z-$ row).

- The third constraint is an equality, so $y_3 = \frac{9}{23}$ (coefficient of a_3 in the $Z-$ row minus M).

The optimal value of the objective function of the dual problem is $W = 15\frac{51}{23} + 5(-\frac{58}{23}) + 10\frac{9}{23} = \frac{565}{23}$ and equals the optimal value of the objective function in the primal problem.

Chapter 6

Integer Programming (IP)

A linear programming problem in which all the variables are required to be integers is called **an integer programming problem (IPP)**. Integer programming is also called **discrete programming**.

Integer Programming Problem

$$\begin{array}{ll}
\max(\min)z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n & \text{[Objective function]} \\
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le (\ge, =)b_1 & \text{[Constraint 1]} \\
\ldots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le (\ge, =)b_m & \text{[Constraint } m\text{]} \\
x_1 \ge 0, \ldots, x_r \ge 0, \;\; r \le n & \text{[Sign restrictions]} \\
x_i - \text{integer}, \; i = 1, \ldots, n &
\end{array}$$

If only some of the variables required to have integer values, this model is referred to as **mixed integer programming (MIP)**. When distinguishing the all-integer problem from this mixed case, we call the former **pure integer programming**. There are problems involving a number of interrelated "'yes-or-no"' decisions. In such decisions, the only possible choices are *yes* and *no*. We can represent such decisions by decision variables that are restricted to just two values, say 0 and 1(zero-one). Thus, the jth yes-or-no decision would be represented by, say, x_j such that

$$x_j = \left\{ \begin{array}{ll} 1 & \text{if decision } j \text{ is yes} \\ 0 & \text{if decision } j \text{ is no.} \end{array} \right.$$

Such variables are called **binary variables**(or 0-1 variables). Problems that contain only binary (0-1)variables are called *Binary Integer Programming Problems (BIP)*.

$$\begin{array}{ll}
\max(\min)z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n & \text{[Objective function]} \\
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le (\ge, =)b_1 & \text{[Constraint 1]} \\
\ldots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le (\ge, =)b_m & \text{[Constraint } m\text{]} \\
x_i \in \{0,1\}, i = 1, \ldots, n & \text{[0-1 or binary variables.]}
\end{array}$$

6.1 FORMULATING INTEGER PROGRAMMING PROBLEMS

Example 1. ***The Cutting Stock(or Trim Loss) Problem.*** *A sawmill produces standard boards which are 10 inches wide and l inches long. It receives orders for 100 boards which are 2 inches wide, 150 boards which are 3 inches wide and 80 boards which are 4 inches wide. All these boards are required to be l inches in length. The sawmill wishes to determine how to meet these orders so as to minimize total waste.*

Waste is defined as any leftover portion of a standard board which cannot be used to meet demand. There are two sources of waste: trim loss and surplus. For example, if a standard board is split into three 3-inch boards, there will be leftovers (trim loss) having a width of 1 inch. Note that this cutting pattern yields three 3-inch boards from each standard board which is cut. Because different cutting patterns can create multiple boards out of standard boards, there is a possibility that a surplus or excess number will be cut. For example, if 155 3-inch boards result from the cutting process and only 150 are required, then the surplus waste is (155-150)×3. All the different ways (patterns) in which a standard board can be split are given in the table:

Pattern	Number of 4-inch boards	Number of 3-inch boards	Number of 2-inch boards	Waste (in inches)
1	2	0	1	0
2	1	2	0	0
3	1	1	1	1
4	1	0	3	0
5	0	3	0	1
6	0	2	2	0
7	0	1	3	1
8	0	0	5	0

Decision variables:

- x_i = *number of standard boards which are to be split according to pattern* i, *where* $i = 1, \ldots, 8$,

- y_1 = *number of excess 4-inch boards,* y_2 = *number of excess 3-inch boards,* y_3 = *number of excess 2-inch boards.*

Model:

$$x_3 + x_5 + x_7 + 4y_1 + 3y_2 + 2y_3 \to \min$$
$$2x_1 + x_2 + x_3 + x_4 - y_1 = 80 \qquad \text{[4-inch boards]}$$
$$2x_2 + x_3 + 3x_5 + 2x_6 + x_7 - y_2 = 150 \qquad \text{[3-inch boards]}$$
$$x_1 + x_3 + 3x_4 + 2x_6 + 3x_7 + 5x_8 - y_1 = 100 \qquad \text{[2-inch board]}$$
$$x_i, y_j \geq 0, \; x_i, y_j - integer, \; i = 1, \ldots, 8, ; j = 1, 2, 3.$$

Example 2. ***Capital Budgeting with Fixed Investment Proposals.*** *(F.S. Hillier, G.J. Liebermen Introduction to Operations Research, Example, pp.479-480.) The company CHC is considering building a new factory in either city A or city B, or perhaps even in both cities. It is also considering building at most one new warehouse, but the choice of location is restricted to a city where a new factory is being built. The net present value - NPV (total profit considering the time value of money) of each of these alternatives together with the capital required for the respective investments, are given in the table:*

Decision number	Yes-or-No decision	Decision variable	NPV $ mln	Capital required
1	Build factory in A?	x_1	$9	$6
2	Build factory in B?	x_2	$5	$3
3	Build warehouse in A?	x_3	$6	$5
4	Build warehouse in B?	x_3	$4	$2

The total capital available is $10000000. The objective is to find the feasible combination of alternatives that maximizes the total net present value.

Decision variables:

$$x_j = \begin{cases} 1 \text{ if decision } j \text{ is YES} \\ 0 \text{ if decision } j \text{ is NO} \end{cases} j = 1, 2, 3, 4.$$

Binary Integer Programming Model: The objective function $Z = $ total net present value: $Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$. The amount of capital expended on the four facilities cannot exceed \$10 million: $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$. The company wants at most one new warehouse: $x_3 + x_4 \leq 1$. The company would consider building a warehouse in a city only if a new factory were also going there:

$$x_3 \leq x_1 \qquad (\text{ if } x_1 = 0 \text{ then } x_3 = 0,)$$
$$x_4 \leq x_2 \qquad (\text{ if } x_2 = 0 \text{ then } x_2 = 0).$$

The final model is as follows:

$$\begin{array}{rcl} 9x_1 + 5x_2 + 6x_3 + 4x_4 & = & Z \to \max \\ 6x_1 + 3x_2 + 5x_3 + 2x_4 & \leq & 10 \\ x_3 + x_4 & \leq & 1 \\ -x_1 \qquad + x_3 & \leq & 0 \\ -x_2 \qquad + x_4 & \leq & 0 \\ x_i \in \{0,1\} \text{ for } i=1,2,3,4. \end{array}$$

Example 3. *The Knapsack Problem.* *Joe C. is going on an overnight hike. There are seven items Joe is considering taking on the trip. The weight of each item and the benefit Joe feels he would obtain from each are listed in the following table:*

Item	Weight	Benefit
1	5	8
2	2	3
3	7	10
4	1	1
5	6	9
6	8	11
7	2	2

Joe's knapsack can hold up to 15 lb of items. For $j = 1, 2, \ldots, 7$, define

$$x_j = \begin{cases} 1 & \text{if Joe takes item } j \text{ on the trip} \\ 0 & \text{otherwise} \end{cases}$$

Then Joe can maximize his total benefit by solving the following binary problem:

$\max Z = 8x_1 + 3x_2 + 10x_3 + x_4 + 9x_5 + 11x_6 + 2x_7$
$5x_1 + 2x_2 + 7x_3 + x_4 + 6x_5 + 8x_6 + 2x_7 \leq 15$ *[Capacity of the knapsack]*
$x_i \in \{0,1\}, \; i = 1, \ldots, 7$

We consider the following additional constraints:

- *Joe has to take item 2 or item 5(at least one item in a group). It suffices to model the alternative $(x_2 = 1) \lor (x_5 = 1)$ by adding the constraint:*

$$x_2 + x_5 \geq 1$$

- *Joe cannot take both item 1 and item 6. We model the condition: $\neg(x_1 = 1 \land x_6 = 1) \equiv (x_1 = 0 \lor x_6 = 0)$ by adding the constraint:*

$$x_1 + x_6 \leq 1$$

- *If Joe takes item 3, then he must also take item 4. Now we model implication:*
  $(x_3 = 1) \Rightarrow (x_4 = 1)$ *by adding the constraint:*

$$x_4 \geq x_3$$

Example 4. ***Fixed-charge Problem.*** *(based on W.L. Winston Operations Research: Applications and Algorithms, Example 8.3, pp.369-372.)*
The firm STYLE is capable of manufacturing three types of products: P_1, P_2 and P_3. The manufacture of each type of product requires the appropriate type of machines. These machines must be rented at the following rates: machines for P_1, \$200 per week; machines for P_2, \$150 per week; machines for P_3, \$100 per week. The manufacture of each type of product also requires the amounts of cloth and labor given in table (6.1). Each week, 150 hours of labor and 160 sq yd of cloth are available.

	Labor (hours)	Cloth (square yards)
P_1	3	4
P_2	2	3
P_3	6	4

Table 6.1: Resource requirements for STYLE

The variable unit cost and selling price for each type of product are shown in table (6.2). Formulate an IPP whose solution will maximize STYLE's profits.

	Sales Price	Variable Cost
P_1	\$12	\$6
P_2	\$8	\$4
P_3	\$15	\$8

Table 6.2: Revenue and Cost for STYLE

- x_i = *number of units of P_i produced each week, for $i = 1, 2, 3$.*

- $y_i \in \{0, 1\}, y_i = 1$ *if product P_i is to be produced and $y_i = 0$ otherwise.*

Model: The objective function is weekly profit(= weekly revenue - weekly variable costs - weekly costs of renting machines)

$$\max Z = 12x_1 + 8x_2 + 15x_3 - (6x_1 + 4x_2 + 8x_3) - (200y_1 + 150y_2 + 100y_3)$$
$$= 6x_1 + 4x_2 + 7x_3 - 200y_1 - 150y_2 - 100y_3$$
$$3x_1 + 4x_2 + 7x_3 \leq 150 [\text{Labor constraint}]$$
$$4x_1 + 3x_2 + 7x_3 \leq 160 [\text{Cloth constraint}]$$
$$x_1 \leq M_1 y_1 (\text{if } x_1 > 0 \text{ then } y_1 = 1)$$
$$x_2 \leq M_2 y_2 (\text{if } x_2 > 0 \text{ then } y_2 = 1)$$
$$x_3 \leq M_3 y_3 (\text{if } x_3 > 0 \text{ then } y_3 = 1)$$
$$x_1, x_2, x_3 \geq 0; y_1, y_2, y_3 \in \{0, 1\}$$

From the Labor and Cloth constraints we obtain: $M_1 = 40, M_2 = 53$ and $M_3 = 25$. The optimal solution is: $Z^{opt} = 75, x_3 = 25, y_3 = 1$.

Example 5. ***Set-Covering Problem.*** *(based on W.L. Winston Operations Research: Applications and Algorithms, Example 8.5, pp.375-376.)*
There are six cities in a county. The county wants to build the minimum number of fire stations needed to ensure that at least one fire station is within 15 minutes (driving time) of each city. The times (in minutes) required to drive between the cities in the county are shown in the following table:

	City 1	City 2	City 3	City 4	City 5	City 6
City 1	0	10	20	30	30	20
City 2		0	25	35	20	10
City 3			0	15	30	20
City 4				0	15	25
City 5					0	14

The county must determine how many fire stations should be built and where they should be located. The cities within 15 minutes of a given city are given in the following table:

Fire station	can serve cities
1	1,2
2	1,2,6
3	3,4
4	3,4,5
5	4,5,6
6	2,5,6

Decision variables:

- $x_i \in \{0,1\}$, $x_i = 1$ *if a fire station is built in city i and 0 otherwise, $i = 1, \ldots, 6$.*

The constraints must ensure that there is a fire station within 15 minutes of each city. Model:

$$
\begin{aligned}
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \to \min &\\
&x_1 + x_2 \geq 1 &[City\ 1]\\
&x_1 + x_2 + x_6 \geq 1 &[City\ 2]\\
&x_3 + x_4 \geq 1 &[City\ 3]\\
&x_3 + x_4 + x_5 \geq 1 &[City\ 4]\\
&x_4 + x_5 + x_6 \geq 1 &[City\ 5]\\
&x_2 + x_5 + x_6 \geq 1 &[City\ 6]\\
&x_i \in \{0,1\},\ i = 1, \ldots, 6 &
\end{aligned}
$$

6.2 ADVANCED MODELING

Example 6. *(based on W.L. Winston Operations Research: Applications and Algorithms, Example 8.6, pp.377-379.)*
The firm DA manufactures three types of autos: compact(TYP1), mid size(TYP2), and large(TYP3). The resources required for, and the profit yielded by, each type of car are given in the following table:

	TYP 1	TYP 2	TYP 3
Steel required (tons)	1.5	3	5
Labor required (hours)	30	25	40
Profit yielded ($)	2000	3000	4000

At present 6000 tons of steel and 60000 hours of labor are available. The firm DA wants to maximize its profit.

Decision variables:

- x_i *- number of cars of type i produced, $i = 1, \ldots, 3$.*

Model:

$$2000x_1 + 3000x_2 + 4000x_3 \to \max$$
$$1.5x_1 + 3x_2 + 5x_3 \leq 6000 \qquad \textit{[Steel usage constraint]}$$
$$30x_1 + 25x_2 + 40x_3 \leq 60000 \qquad \textit{[Labor usage constraint]}$$
$$x_1, x_2, x_3 \geq 0 \textit{ and integer}$$

Let us consider the following additional restrictions:

1. *If any cars of a given type are produced, then at least 1000 cars of that type must be produced. Thus we must have that $x_i \leq 0$ or $x_i \geq 1000$ for $i = 1, 2, 3$. We should model the alternative:$(x_1 \leq 0) \vee (x_1 \geq 1000)$ as follows: We introduce a binary variable $y_1 \in \{0, 1\}$ and add two constraints:*

$$x_1 \leq My_1$$
$$1000 - x_1 \leq M(1 - y_1)$$

   where M is a sufficiently large number. If $y_1 = 0$ then $x_1 \leq 0$. And if $y_1 = 1$ then $1000 - x_1 \leq 0$ which gives $x_1 \geq 1000$. Therefore this ensures that at least one of these constraint is satisfied.

2. *If the number of cars of type 3 produced is greater than 500 than the production of cars of type 2 should be less or equal to 100. Now we want to model a logical implication (**if-then constraint**): $(x_3 > 500) \Rightarrow (x_2 \leq 100)$. Applying the following identity: $(p \Rightarrow q) \equiv (\sim p \vee q)$ we obtain $(x_3 > 500) \Rightarrow (x_2 \leq 100) \equiv (x_3 \leq 500) \vee (x_2 \leq 100) \equiv (x_3 - 500 \leq 0) \vee (x_2 - 100 \leq 0)$. We introduce a binary variable $y_2 \in \{0, 1\}$ and include the following constraints (see 6.1) in the formulation:*

$$x_3 - 500 \leq My_2$$
$$x_2 - 100 \leq M(1 - y_2)$$

   where M is a large positive number. If $x_3 > 500$, then the first constraint can be satisfied only if $y_2 = 1$. Then from the second constraint we obtain $x_2 \leq 100$. If $x_3 \leq 500$, then $y_2 = 0$ and the value of x_2 is unrestricted (may be any number).

EITHER-OR-CONSTRAINTS

In the general case, we want to model the alternative:

$$f(x_1, \ldots, x_n) \leq 0 \vee g(x_1, \ldots, x_n) \leq 0$$

We wish to ensure that <u>at least one of two constraints</u> is satisfied. We model such a constraint by introducing a binary variable $y \in \{0, 1\}$ and changing the constraints as follows:

$$f(x_1, \ldots, x_n) \leq My$$
$$g(x_1, \ldots, x_n) \leq M(1 - y) \qquad (6.1)$$

where M is sufficiently large number to ensure that $f(x_1, \ldots, x_n) \leq M$ and $g(x_1, \ldots, x_n) \leq M$.

IF-THEN CONSTRAINTS

In many situations we wish to model the implication

$$f(x_1, \ldots, x_n) > 0 \Rightarrow g(x_1, \ldots, x_n) \leq 0.$$

We can use an equivalent condition:

$$f(x_1, \ldots, x_n) \leq 0 \vee g(x_1, \ldots, x_n) \leq 0.$$

and include constraints ([6.1]) in the formulation.

If we wish to model the implication:

$$f(x_1, \ldots, x_n) > 0 \Rightarrow g(x_1, \ldots, x_n) \geq 0$$

then we use an equivalent condition:

$$f(x_1, \ldots, x_n) \leq 0 \vee -g(x_1, \ldots, x_n) \leq 0,$$

and include the following constraints in the formulation:

$$
\begin{array}{rcl}
f(x_1, \ldots, x_n) & \leq & My \\
-g(x_1, \ldots, x_n) & \leq & M(1-y) \\
y & \in & \{0, 1\}.
\end{array}
$$

Example 7. ***The Transportation Problem (TP)***
*The general transportation problem is concerned with distributing any commodity from a group of supply centers, called **sources**, to any group of receiving centers, called **destinations**, in such a way as to minimize the total cost of distribution. Each source has a fixed supply of units, and this entire supply must be distributed to the destinations. Let s_i denote the number of units supplied by source i, for $i = 1, 2, \ldots, m$. Each destination has a fixed demand for units, and this entire demand must be received from the sources. Let d_j denote the number of units received by destination j, for $j = 1, 2, \ldots, n$. A transportation problem will have a feasible solution if and only if*

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

*If the problem violates this assumption, it is possible to reformulate the problem by introducing a dummy destination or dummy source to take up the slack between the actual amounts and maximum amounts to be distributed. The cost of distributing units from any particular source to any particular destination is assumed to be directly proportional to the number of units distributed, i.e. this cost is just the unit cost of distribution times the number of units distributed. The unit cost of distributing of a commodity from source i to destination j is denoted by c_{ij}. All the data for $\mathbb{TP}$ are included in the following parameter table ([6.3]): **Decision***

| Source | Cost Per Unit Distributed | | | | Supply |
| | Destination | | | | |
	1	2	$\cdots$	n	
1	c_{11}	c_{12}	$\cdots$	c_{1n}	s_1
2	c_{21}	c_{22}	$\cdots$	c_{2n}	s_2
$\vdots$		$\ldots\ldots\ldots\ldots$			$\vdots$
m	c_{m1}	c_{m2}	$\cdots$	c_{mn}	s_m
Demand	d_1	d_2	$\cdots$	d_n	

Table 6.3: Parameter table for the transportation problem

variables:

- x_{ij} =*the number of units to be distributed from source i to destination j, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.*

Model:

$$Minimize\ Z\ =\ \sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}$$

$$\sum_{j=1}^{n}x_{ij}\ =\ s_i\ for\ i\text{=}1,2,...,m,$$

$$\sum_{i=1}^{m}x_{ij}\ =\ d_j\ for\ j\text{=}1,2,...,n,$$

$$x_{ij}\ \geq\ 0,\ for\ all\ i\ and\ j.$$

Integer solutions property: *For transportation problems, where all the s_i and d_j have integer values, all the basic variables in every basic feasible solution (including an optimal basic solution) also have integer values.*

Example 8. The Traveling Salesman Problem (TSP). *A salesman has to set out from home to visit a number of customers before finally returning home. The problem is to find the order in which he should visit all the customers to minimize the total distance covered. The traveling salesman problem can be formulated as an integer programming problem as follows: Suppose the cities to be visited are numbered $1, 2, \ldots, n$. Any solution to the problem will be referred to as a* **tour.** *Define the $0-1$ integer variables x_{ij} as follows:*

$$x_{ij} = \begin{cases} 1 & \text{if the tour goes from i to j directly.} \\ 0 & \text{otherwise.} \end{cases}$$

and extra continuous variables u_i for $i = 2, \ldots, n$ are introduced. The following interpretation can be applied to the variables u_i: $u_i = j$ if i is the j-th city visited.
    Model:

$$Minimize Z\ =\ \sum_{i\neq j}^{n}c_{ij}x_{ij} \tag{6.2}$$

$$\sum_{j=1,\ j\neq i}^{n}x_{ij}\ =\ 1\ for\ i=1,\ldots,n. \tag{6.3}$$

$$\sum_{i=1,\ i\neq j}^{n}x_{ij}\ =\ 1\ for\ j=1,\ldots,n. \tag{6.4}$$

$$x_{ij}\ \in\ \{0,1\}\ i,j=1,\ldots,n;i\neq j. \tag{6.5}$$

$$u_i - u_j + (n-1)x_{ij}\ \leq\ n-2\ for\ i,j=2,\ldots,n;i\neq j. \tag{6.6}$$

$$1\leq u_i\ \leq\ n-1\ for\ i\text{=}2,\ldots,n. \tag{6.7}$$

Conditions (3) states that exactly one city must be visited immediately after city i. Conditions (4) states that exactly one city must be visited immediately before city j. Constraint (6) is added to the model to prevent subtours ever arising.

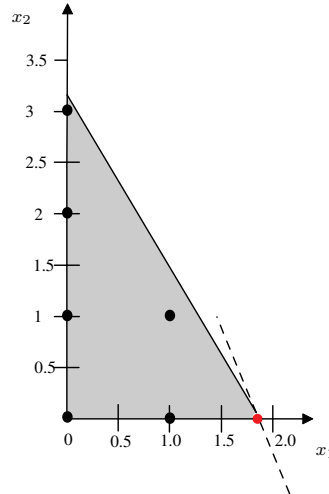Special algorithms have been developed for the traveling salesman problem.

6.3   NAIVE METHODS FOR SOLVING IPP

1. Apply the simplex algorithm to the LPP obtained by omitting all the integer constraints and then round the non integer values to integers in the resulting

solution. We consider an example:

$$\max z = 21x_1 + 11x_2$$
$$7x_1 + 4x_2 \leq 13$$
$$x_1, x_2 \geq 0, \ x_1, x_2 \text{ integer}$$

There are six integer feasible solutions (x_1, x_2): (0,0), (0,1), (0,2), (0,3), (1,0) and (1,1). The objective function attains its maximum at the point (0,3). Omitting the integer constraints on the variables we obtain the optimal solution (non integer) at the point (13/7,0). Rounding this result, we obtain the point (2,0), which is not feasible. Rounding down we receive point(1, 0), which is not optimal. It may be far from the optimal in terms of the value of objective function.

2. Perform exhaustive enumeration: Check each solution for feasibility, and if it is feasible, calculate the value of the objective function. Choose the best one. We consider the knapsack problem:

$$\max Z = \sum_{i=1}^{n} c_i x_i$$
$$\sum_{i=1}^{n} w_i x_i \leq W$$
$$x_i \in \{0, 1\}, \ i = 1 \ldots n$$

We need to generate and evaluate 2^n solution to this problem. Assuming that it takes 10^{-6} s. to evaluate one solution, we get that for $n = 50$ the computer time required will be 35 years and for $n = 60$ this time will be approximately 36 558 years. So algorithms performing exhaustive enumeration are not effective.

6.4 THE BRANCH-AND-BOUND METHOD FOR SOLVING PURE INTEGER PROGRAMMING PROBLEMS

Remark 1. *There is no an efficient algorithm for solving general IPP.*

Definition 4. *The linear programming problem obtained by omitting all the integer or 0-1 constraints on the variables is called the **LP relaxation** of the IPP.*

Example 9.

$$\max Z = 21x_1 + 11x_2$$
$$7x_1 + 4x_2 \leq 13$$
$$x_1, x_2 \geq 0$$
$$x_1, x_2 \ integer$$

$$\max Z_R = 21x_1 + 11x_2$$
$$7x_1 + 4x_2 \leq 13$$
$$x_1, x_2 \geq 0$$

We introduce the following notions:

- Z^* - the maximal value of the objective function for the IPP,

- Z_R^*- the maximal value of the objective function fot the LP relaxation of the IPP.

Property 4. *For a maximum problem, the following condition is satisfied $Z_R^* \geq Z^*$, so the LP relaxation gives an upper bound on the value of the objective function for the IPP.*

We can solve the LP relaxation using the simplex algorithm. This will be used in the branch-and-bound algorithm.

Example 10. *Solve the problem:*

$$\max Z = 8x_1 + 5x_2$$
$$6x_1 + 10x_2 \leq 45$$
$$9x_1 + 5x_2 \leq 45$$
$$x_1, x_2 \geq 0, \ x_1, x_2 \ integer$$

The branch-and-bound method begins by solving the LP relaxation of the IPP. Using, for example, the simplex algorithm or graphical method we obtain the optimal solution: $x_1 = 3.75$, $x_2 = 2.25$, $Z_R^* = 41.25$.
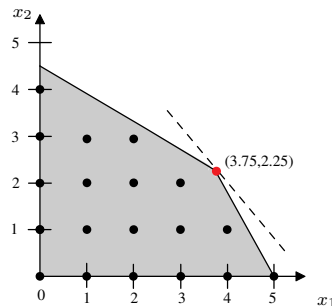

Remark 2. *If we solve the LP relaxation of a pure IPP and obtain a solution in which all the variables are integer, then the optimal solution to the LP relaxation is also the optimal solution to the IPP.*

The solution obtained here is not feasible (not all the variables are integer). We choose a variable that is fractional with the largest coefficient in the objective function i.e. x_1. Each point in the feasible region for this problem must have either $x_1 \leq 3$ or $x_1 \geq 4$. We branch according to the variable x_1 and create the following two additional subproblems (called subproblem 1 and subproblem 2. The Original problem is called problem 0):
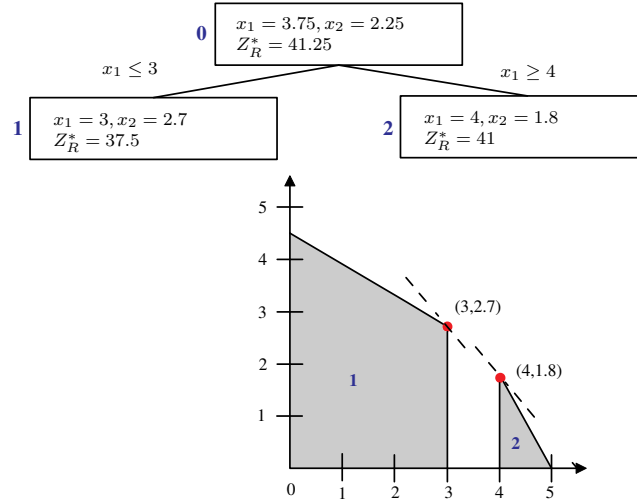
A display of all the subproblems that have been created is called a tree. Each subproblem corresponds to a node of the tree, and each line connecting two nodes of the tree is called an arc. The constraints associated with any node of the tree are the constraints of the LP relaxation plus the constraints associated with the arcs leading from problem 0 to that node. Solving both relaxations of the newly formed subproblems using the simplex method yields infeasible solutions(non-integer). We now choose the subproblem (node) with the greater optimal value of the objective value for the LP relaxation. This is subproblem 2 (node 2). The variable $x_2 = 1.8$ is fractional in the optimal solution to the LP relaxation. We partition the feasible region for subproblem 2 into those points having $x_2 \leq 1$ or $x_2 \geq 2$. This creates the two subproblems 3 and 4 (nodes 3 and 4). Subproblem 4 is infeasible (has no feasible solutions), thus it cannot yield the optimal integer solution. This subproblem (node 4) will not be used for further branching - we say that the subproblem(or node) is closed (or **fathomed**).
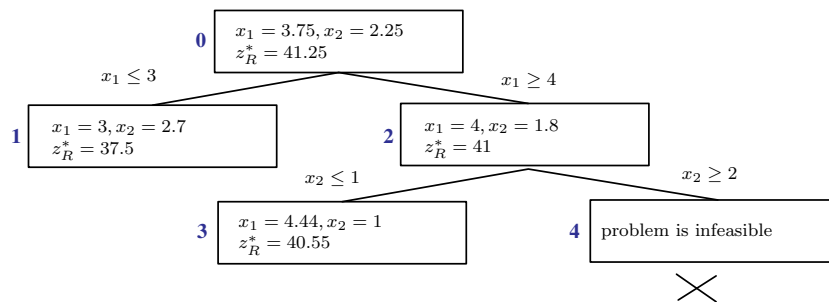
So we choose to partition subproblem 3 (node 3) by branching according to the variable x_1. This yields two subproblems 5 and 6. Together, subproblems 5 and 6 include all the integer points that were included in the feasible region for subproblem 3. Solving these two subproblems using the simplex algorithm, we obtain optimal solutions in which all the variables have integer values - such integer solutions are called **candidate solutions**. Since a candidate solution may be optimal, we must keep a candidate solution until a better feasible integer solution (if any exists)

is found. We close these nodes. Now we have feasible solution (integer) $x_1 = 5, x_2 = 0, Z = 40$. We still have to investigate the open subproblem 1. But the upper bound for this subproblem equals $Z_R^* = 37.5$ is less then 40. Thus any feasible solution of subproblem 1 is no better than the current integer solution $x_1 = 5, x_2 = 0, Z = 40$. The node corresponding to subproblem 1 is thus closed(it is fathomed). Now all nodes are closed (fathomed) and the optimal integer solution is $x_1 = 5, x_2 = 0, Z = 40$.
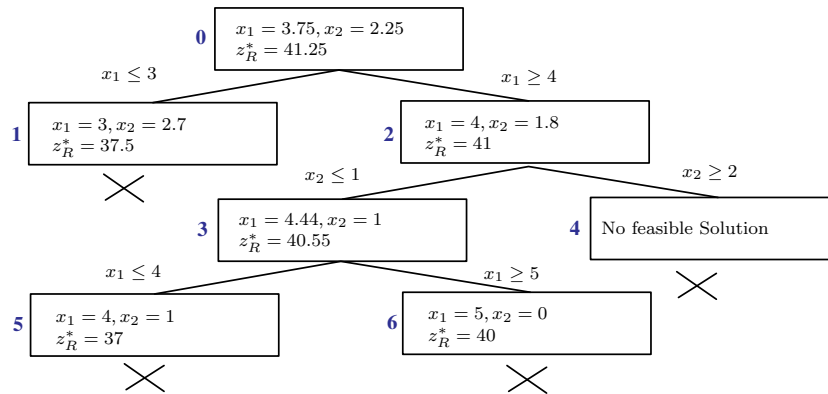
6.5 THE BRANCH-AND-BOUND METHOD FOR SOLVING MIXED INTEGER PROGRAMMING PROBLEMS

To solve a mixed integer programming problem (MIPP) using branch-and-bound, we modify the method for pure IP by branching only according to variables that are required to be integers. *Example.* Solve the problem:

$$\max z = 2x_1 + x_2$$
$$5x_1 + 2x_2 \leq 8$$
$$x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0, \ x_1 \text{ integer}$$

In the mixed integer programming problem some variables are required to be integers and others are allowed to be either integers or non-integers. We begin the branch-and-bound method by solving the LP relaxation. The optimal solution is $x_1 = 2/3, x_2 = 7/3, Z_R^* = 11$. Since only x_2 is allowed to be fractional, we must branch according to x_1. The branch-and-bound tree is as follows:
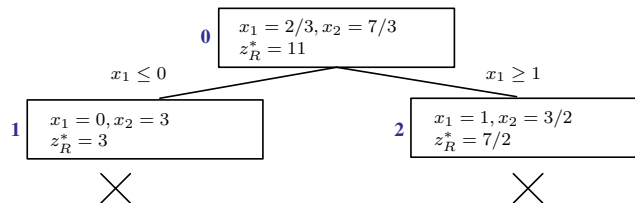
The optimal solution $x_1 = 1, x_2 = 3/2, Z^* = 7/2$ is situated at the node 2.
Remarks: The node (subproblem) k is closed or fathomed (eliminated from further partition) if:

1. The solution of the LP relaxation for node (subproblem) k is feasible (yields a candidate solution, i.e. the appropriate variables are integers).

2. The subproblem for node k is not consistent (has no feasible solutions).

3. An integer solution (candidate solution) has been found such that the value of the objective function is equal or greater than optimal objective value of the LP relaxation for node(subproblem) k. In this case, the subproblem k cannot contain a better solution.

6.6   BRANCH-AND-BOUND ALGORITHM FOR KNAPSACK PROBLEM

Let us consider the following problem:

$$\max z = \sum_{i=1}^{n} c_i x_i$$
$$\sum_{i=1}^{n} w_i x_i \leq W$$
$$x_i \in \{0, 1\}, \ i = 1 \ldots n$$

where c_i - the benefit obtained if item i is chosen, w_i - weight of i-th item, W - capacity of knapsack.

Example. Solve the problem:

$$\max z = 5x_1 + 3x_2 + 6x_3 + 6x_4 + 2x_5$$
$$5x_1 + 4x_2 + 7x_3 + 6x_4 + 2x_5 \leq 15,$$
$$x_1, .., x_5 \in \{0, 1\}$$

The LP relaxation of the problem is as follows:

$$\max z^R = 5x_1 + 3x_2 + 6x_3 + 6x_4 + 2x_5$$
$$5x_1 + 4x_2 + 7x_3 + 6x_4 + 2x_5 \leq 15,$$
$$0 \leq x_i \leq 1, \ i = 1, \ldots, 5$$

This problem is called *the continuous knapsack problem.* Now it is allowed to take part of an item. The LP relaxation is solved using the following greedy algorithm: We begin by computing the $\frac{c_i}{w_i}$ ratios and ordering the variables from best

| i	c_i	w_i	c_i/w_i	Ranking
1	5	5	1	1
2	3	4	3/4	5
3	6	7	6/7	4
4	6	6	1	2
5	2	2	1	3

Table 6.4: Ordering items from best to worst

to worst(the items 1,4 and 5 can be ordered in an arbitrary way). First we put the best item in the knapsack. Then put the second-best item in the knapsack. Continue in this fashion until the best remaining item not fit into the knapsack. Then fill the knapsack with as much of this item as possible (this is called greedy method). At the beginning, the capacity of the knapsack is $W = 15$. First we put item 1 in the knapsack (the remaining capacity of the knapsack is $W = 10$), then we put item 4 ($W = 4$), next we put item 5 ($W = 2$) and we put $\frac{2}{7}$ of item

3 (we have now used all the capacity $W = 0$). We have obtained the solution $x_1 = 1, x_2 = 0, x_3 = \frac{2}{7}, x_4 = 1, x_5 = 1$ and $Z_R^* = 14\frac{5}{7}$. This solution is not feasible (not all the variables are binary) - the variable $x_3 = \frac{2}{7}$ is fractional. We choose to partition problem 0, by branching according the variable x_3. This yields two subproblems 1 and 2:

1. Problem 1 - we set $x_3 = 0$ (we consider the LP relaxation without item 3 but the capacity of knapsack is $W = 15$). The optimal solution of the LP relaxation is $x_1 = 1$, $x_2 = \frac{1}{2}$, $x_3 = 0$, $x_4 = 1$, $x_5 = 1$ i $Z_R^* = 14\frac{1}{2}$.

2. Problem 2 - we set $x_3 = 1$ (we first include item 3 in the knapsack and the remaining capacity of the knapsack 15-5=10 is filled using the greedy method). We get the optimal solution of the LP relaxation: $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = \frac{1}{2}$, $x_5 = 0$ and $Z_R^* = 14$.

These problems have no-integer solutions. So we choose the subproblem (the node of the branch-and-bound tree) with the largest upper bound z_R^* (the optimal value of the LP relaxation). This is subproblem 1 and it will be further partitioned. The full branch-and-bound tree for this problem is given by the following picture:
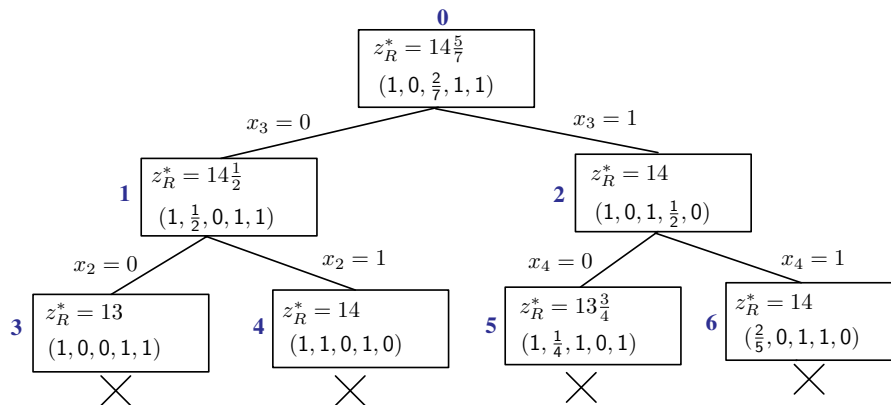
Nodes 3 and 4 are closed because they yield the candidate solutions. After obtaining the candidate solution in node 4 we can close the node 2 because there are no integer solutions with objective function value greater than 14. The optimal solution is in node 4: items 1,2 and 4 have to be put in the knapsack.

Chapter 7

Decision Analysis and Decision Trees

7.1 DECISION CRITERIA UNDER UNCERTAINTY

So far we have focused mainly on decision making when the consequences of alternative decisions are known with a reasonable degree of certainty. Such problems are called problems under conditions of certainty. Now we will study situations in which decisions are made in an uncertain environment. Decision analysis provides a framework and methodology for rational decision making when the outcomes of a decision are uncertain. First we will formulate a general framework for decision making: The decision maker must choose an **action** from a set of possible actions. This set contains all the *feasible alternatives* (or decision alternatives) under consideration for how to proceed with the problem considered. This choice of action must be made in the face of uncertainty, because the outcome will be affected by random factors that are outside the control of the decision maker. These random factors determine what situation will exist when the action is executed. Each of these possible situations is referred to as a possible **state of nature.** For each combination of an action and state of nature, the decision maker knows, what the resulting payoff(or any measure which reflects gain or loss) would be. The **payoff** is a quantitative measure of the value to the decision maker of the consequences of the outcome. A **payoff table** is commonly used to define the payoff v_{ij} for each combination of an action (decision alternative) a_i, $i = 1, \ldots, m$ and a state of nature $\theta_j, j = 1, \ldots, n$. Such a table is illustrated in table 7.1:

Actions	States of nature				
	θ_1	$\cdots$	θ_j	$\cdots$	θ_n
a_1	v_{11}	$\cdots$	v_{1j}	$\cdots$	v_{1n}
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
a_i	v_{i1}	$\cdots$	v_{ij}	$\cdots$	v_{in}
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
a_m	v_{m1}	$\cdots$	v_{mj}	$\cdots$	v_{mn}

Table 7.1: Payoff table

Let us consider the following example:

Example 11. *(Based on W.L. Winston Operations Research: Applications and*

Algorithms, Example 12.1, pp. 569-750.) News vendor Phyllis sells newspapers, and each day she must determine how many newspapers to order. Phyllis pays a company $0.20 for each paper and sells the papers for $0.25 each. Newspapers that are unsold at the end of the day are worthless. Phyllis knows that each day she can sell between 6 and 10 papers, with each possibility being equally likely. The set of possible actions (papers purchased) is $\{a_1(6), a_2(7), a_3(8), a_4(9), a_5(10)\}$ and the set of states of nature(papers demanded) is $\{\theta_1(6), \theta_2(7), \theta_3(8), \theta_4(9), \theta_5(10)\}$. If Phylis purchases i papers and j papers are demanded, then i papers are purchased at a cost of $0.20 and $\min\{i, j\}$ are sold for $0.25 each. Thus she earns a net profit(payoff) of v_{ij}:

$$v_{ij} = \quad 0.25(5+i) - 0.20(5+i) = 0.05(5+i) \quad (i \leq j) \tag{7.1}$$

$$v_{ij} = \quad\quad\quad 0.25(5+j) - 0.20(5+i) \quad\quad\quad (i \geq j) \tag{7.2}$$

where $i, j = 1, \ldots, 5$. The payoff table (the values v_{ij} in dollars) for this problem is given in table 11:

Actions	States of nature(papers demanded)					Minimum
(Papers purchased)	$\theta_1(6)$	$\theta_2(7)$	$\theta_3(8)$	$\theta_4(9)$	$\theta_5(10)$	
$a_1(6)$	0.30	0.30	0.30	0.30	0.30	0.3
$a_2(7)$	0.10	0.35	0.35	0.35	0.35	0.1
$a_3(8)$	-0.10	0.15	0.40	0.40	0.40	-0.1
$a_4(9)$	-0.30	-0.05	0.20	0.45	0.45	-0.3
$a_5(10)$	-0.50	-0.25	0	0.25	0.50	-0.5

Table 7.2: Payoff table for Phyllis

Why did we not consider the possibility that Phyllis would order 1,2,3,4,5, or more than 10 papers? Answering this question involves the idea of a dominated action (alternative).

Definition 5. *An action (alternative) a_i is **dominated** by action $a_{i'}$ if for all states of nature $\theta_j, j = 1, \ldots, n$, $v_{ij} \leq v_{i'j}$ and for some state of nature $\theta_{j'}$ $v_{ij'} < v_{i'j'}$.*

If action a_i is dominated, then in no state of nature is a_i better than $a_{i'}$, and in at least one state of nature a_i is inferior to $a_{i'}$. Thus, if action a_i is dominated, there is no reason to choose a_i ($a_{i'}$ would be a better choice). From the payoff table, we see that none of the actions $\{a_1, a_2, a_3, a_4, a_5\}$ are dominated. We assume that the payoff table for a decision problem contains only non-dominated alternatives (actions).

We now consider four decision criteria that can be used to choose an action. These criteria differ in how conservative the decision maker is, in the face of uncertainty. We assume that the payoff table is given for the problem, but the decision maker has no information about the relative likelihood of the possible states of nature.

Maximin (or Minimax) Payoff Criterion(Wald 1950) For each alternative, determine the worst case (smallest payoff if v_{ij} is a gain (e.g. profit) and largest payoff if payoff is loss (e.g. cost)). The *maximin(minimax)* criterion chooses the alternative with the best of the worst case outcome. If v_{ij} is loss (cost), then we select the action that corresponds to the *minimax* criterion

$$\min_{a_i} \left\{ \max_{\theta_j} v_{ij} \right\},$$

and if v_{ij} is a gain (profit), then we select the action that corresponds to the *maximin* criterion

$$\max_{a_i} \left\{ \min_{\theta_j} v_{ij} \right\}.$$

For the problem considered*maximin* criterion recommends ordering 6 papers i.e. choosing action a_1.

Hurwicz Criterion(1951) This criterion takes into account the attitudes of the decision maker in the face of uncertainty. Fix the value of the parameter α where $0 \leq \alpha \leq 1$, and assume that v_{ij} represents a gain (profit). The Hurwicz criterion chooses the action a_k so that the following condition is fulfilled:

$$\alpha o_k + (1 - \alpha) s_k = \max_{a_i} \{ \alpha o_i + (1 - \alpha) s_i \},$$

where $s_i = \min_{a_j} \{ v_{ij} \}$, $o_i = \max_{a_j} \{ v_{ij} \}$ and α is a given coefficient. The parameter α is called the **index of optimism(or pessimism)**. If $\alpha = 0$ then we obtain *the maximin* criterion (the most conservative) and if $\alpha = 1$ we obtain *maximax* criterion (the most optimistic). Thus the subjective preferences of a decision maker in environments characterized by risk could be described by this coefficient. For the example considered if $\alpha = 0.5$ (the absence of strong feeling regarding optimism and pessimism) the Hurwicz criterion recommends action a_1. If v_{ij} represents a loss, then the choise of the optimal action a_k is changed to

$$\alpha s_k + (1 - \alpha) o_k = \min_{a_i} \{ \alpha s_i + (1 - \alpha) o_i \}.$$

Minimax Regret Criterion(Savage 1951) We construct the new regret matrix r_{ij} as follows:

$$r_{ij} = \begin{cases} \max_{a_k} \{ v_{kj} \} - v_{ij} \text{ if } v_{ij} \text{ is a gain} \\ v_{ij} - \min_{a_k} \{ v_{kj} \} \text{ if } v_{ij} \text{ is a loss.} \end{cases}$$

The minimax regret criterion chooses an action by applying the minimax criterion to the regret matrix. In other words, the minimax regret criterion attempts to avoid disappointment over what might have been. For Phyllis's problem, the regret matrix r_{ij} is given in table 7.1. The minimax regret

Actions	States of nature(papers demanded)				
(Papers purchased)	$\theta_1(6)$	$\theta_2(7)$	$\theta_3(8)$	$\theta_4(9)$	$\theta_5(10)$
$a_1(6)$	0	0.05	0.10	0.15	0.20
$a_2(7)$	0.20	0	0.05	0.10	0.15
$a_3(8)$	0.40	0.20	0	0.05	0.10
$a_4(9)$	0.60	0.40	0.20	0	0.05
$a_5(10)$	0.80	0.60	0.40	0.20	0

Table 7.3: Regret matrix for Phyllis's problem

criterion recommends ordering 6 or 7 papers.

Laplace Criterion This criterion is based on the **principle of insufficient reason**. Because the probability of occurrence of these states of nature are not known, there is no reason to assume that this probabilities are different. So we use the optimistic assumption that all the states are equally likely to occur

- that is, $Pr\{\theta_1\} = Pr\{\theta_2\} = \cdots = Pr\{\theta_n\} = \frac{1}{n}$. We choose the action a_k with the best mean payoff:

$$\sum_{j=1}^{n} \frac{1}{n} v_{kj} = \max_{i=1}^{m} \{ \sum_{j=1}^{n} \frac{1}{n} v_{ij} \}.$$

If v_{ij} represents a loss, then minimization replaces maximization.

7.2   DECISION CRITERIA UNDER RISK

7.2.1   DECISION MAKING WITHOUT EXPERIMENTATION

The decision maker generally will have some information that should be taken into account about the relative likelihood of the possible states of nature. Such information can usually be translated into a probability distribution referred to as a **prior distribution**. The probability for the respective states of nature provided by the prior distribution are called **prior probabilities**. Let us consider a new example.

Example 12. *(F.S. Hiellier, G.J. Lieberman Operations Research, Prototype Example, pp. 681.)*
The Oil&Gas Co. owns land that may contain oil. A consulting geologist believes that there is 1 chance in 4 (probability 0.25) of oil. Another company has offered to purchase the land for $90000. However, the Oil&Gas Co. is considering holding the land in order to drill for oil itself. The cost of drilling is $100000. If oil is found, the resulting expected revenue will be $800000, so the company's profit (after deducting the cost of drilling) will be $700000. A loss of $100000 will be incurred if the land is dry (no oil). The payoff table for this problem is given in table 7.4.

	State of nature	
Alternative	θ_1(Oil)	θ_2(Dry)
a_1(Drill for oil)	$700000	-$100000
a_2(Sell the land)	$90000	$90000
Probability	0.25	0.75

Table 7.4: Payoff table for the Oil&Gas Co.

There are two criteria that can be used to define an optimal action:

The Maximum Likelihood Criterion Identify the most likely state of nature (the one with the largest probability). For this state of nature, find the action with the maximum payoff. Choose this action. In the example considered the optimal action is a_2 (Sell the land).

Bayes's Decision Rule Calculate the expected value of the payoff for each of the possible actions. Choose the action with the maximum expected payoff. In the considered example expected values for actions a_1, a_2 are respectively $EV(a_1) = 100000, EV(a_2) = 90000$. Since 100000 is larger than 90000, the optimal action is to drill for oil.

For a decision problem under risk, the important notion of the Expected Value of Perfect Information, abbreviated EVPI, is introduced. Suppose now that an experiment could definitely identify what the true state of nature is, thereby providing

"perfect" information. Whichever state of nature is identified, you choose the action with the best payoff for that state. We do not know in advance which state of nature will be identified, so a calculation of the expected payoff with perfect information requires weighting the maximum (minimum if v_{ij} is a cost) payoff for each state of nature by the prior probability of that state of nature. For example, for the Oil&Gas Co. the probability (prior probability) of the states of nature are $Pr(\theta_1) = p_1 = 0.25$ and $Pr(\theta_2) = p_2 = 0.75$,

The expected value with perfect information = 0.25 (700000)+ 0.75(900000)=242500. The EVPI is calculated as

EVPI=Expected value with perfect information - expected value without perfect information.

In the example considered, we have:

$$EVPI = 242500 - 100000 = 142500.$$

So if the cost of experimentation (a seismic survey) is \$30000, it may be worthwhile to proceed with the seismic survey to get better information about the probabilities of the possible states of nature (posterior probabilities).

Example 13. *Bob Warren is the manager of the Butterfly Grocery Store. He needs to replenish his supply of strawberries. His regular supplier can provide as many cases as he wants. However, because these strawberries are already very ripe, he will need to sell them tomorrow and discard any that remain unsold. Bob estimates that he will be able to sell 10, 11, 12 or 13 cases tomorrow. He can purchase the strawberries for \$3 per case and sell them for \$8 per case. Bob now needs to decide how many cases to purchase. He has checked the store's records on daily sales of strawberries. On this basis, he estimates that the prior probabilities are 0.2, 0.4, 0.3 and 0.1 for being able to sell 10, 11, 12 and 13 cases of strawberries tomorrow. Let a_i represent the action of purchasing $9+i(i=1,\ldots,4)$ cases of strawberries and θ_j represent the state of nature that daily demand will be $9 + j(j = 1,\ldots 4)$ cases. Bob earns a net profit (payoff) of v_{ij}:*

$$v_{ij} = \quad 50 + 5(i - 1) \qquad (i \leq j) \qquad (7.3)$$
$$v_{ij} = \quad 5(9 + j) - 3(j - i) \quad (i > j), \qquad (7.4)$$

where $i, j = 1,\ldots, 4$. The payoff table (the values v_{ij} in dollars) for Bob's decision problem is given in table 7.5. The value $EV(a_i)$ is the expected value of alternative a_i, i.e. $EV(a_i) = Pr\{\theta_1\}v_{i1} + Pr\{\theta_2\}v_{i2} + Pr\{\theta_3\}v_{i3} + Pr\{\theta_4\}v_{i4} = 0.2v_{i1} + 0.4v_{i2} + 0.3v_{i3} + 0.1v_{i4}$. Since the maximum expected value is obtained for alternative a_3,

v_{ij}(Profit)	θ_1	θ_2	θ_3	θ_4	$EV(a_i)$
a_1	50	50	50	50	50
a_2	47	55	55	55	53.4
a_3	44	52	60	60	53.6
a_4	41	49	57	65	51.4
Prior probability	0.2	0.4	0.3	0.1	

Table 7.5: Payoff (profit in dollars) table for the Bob problem

the optimal alternative for Bob, according to Bayes' Decision Rule, is to purchase 12 cases of strawberries. He then obtains his maximum expected payoff (profit of \$53.6).

7.2.2 DECISION TREE

A decision tree is another method of analysing decision making problems under risk. These trees are especially helpful when a sequence of decisions must be made. The

definition and application of decision trees will be illustrated by an example.

Example 14. *The investor Mic Risky has just purchased a textile factory and now he is considering three possible decisions:*

1. *Expand the factory to produce army uniforms.*

2. *Still produce the same cotton textiles (Status Quo), but there are a lot of competitors.*

3. *Sell the factory immediately.*

In the case of the first and second alternatives the factory will be sold after one year. The profit will depend on the conditions on the market, which are either **good** *(state of nature θ_1) or* **poor** *(state of nature θ_2). Mic estimates that the probabilities of good and poor market conditions are 0.7 and 0.3, respectively. Payoff table 7.6 summarizes the data for Mic's decision problem.*

| | State of nature | |
| | Good conditions | Poor conditions |
Actions	on the market	on the market
Expand	$800000	$500000
Status quo	$1300000	$-150000
Sell	$320000	$320000
Prior probability	0.7	0.3

Table 7.6: Payoff table for Mic's problem

The problem under consideration could be represented as a decision tree(see Figure 7.1) in which three kinds of nodes (also referred to as forks) are distinguished:

A decision node, represented by a square, which indicates that a decision needs to be made at that point in the process.

A chance node, represented by a big circle, indicates that a random event(state of nature) occurs at that point.

An end(or terminal) node, represented by a small circle.

The three branches emanating from decision node 1 represent the three possible choices (decision alternatives): Expand, Status quo and Sell. Next, the branches emanating from each chance node (2,3 and 4) correspond to the two possible states of nature(good conditions and poor conditions on the market). The number in parentheses along the branch is the probability of a random event called the prior probability of a state of nature. The resulting total payoff is given to the right of each terminal node, evaluated expected value of a given alternative, is given at the corresponding node e.g. chance node 2: $EV(Expand) = (0.7)800000 + (0.3)500000 = 710000$, chance node 3:$EV$(Status quo) $= (0.7)1700000 - (0.3)150000 = 865000$, chance node 4: $EV(Sell) = (0.7)320000 + (0.3)320000 = 320000$. From this decision tree, we can see that the optimal policy for Mic is Status quo, which gives him a maximal expected profit of $865000.
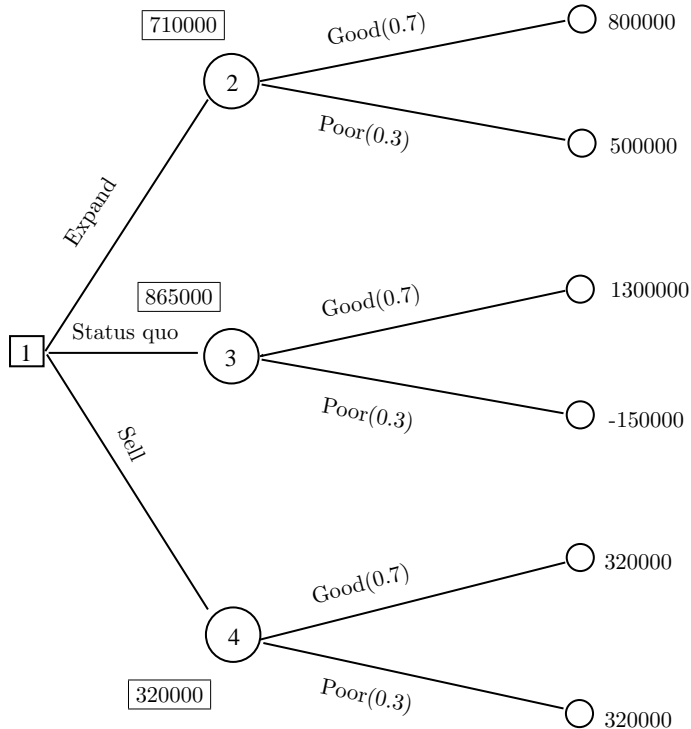
Figure 7.1: Decision tree for Mic's problem

7.2.3 DECISION MAKING WITH EXPERIMENTATION

Now we consider the situation when additional testing(experimentation) can be done to improve the preliminary estimates of the probabilities of the respective states of nature given by the prior probalities. These improved estimates are called **posterior probabilities**. Let us assume that in the problem considered Mic engaged a consulting firm to obtain better estimates of the market situation in the future. The firm formulates a report which will be positive (P) - indicating good future conditions on the market or negative (N) - indicating poor conditions on the market. Based on past experience, if there are good (g) or poor (p) conditions on the market, the conditional probabilities are as follows:

$$Pr\{P/g\} = 0.7, \ Pr\{N/g\} = 0.3; \ Pr\{P/p\} = 0.2, \ Pr\{N/p\} = 0.8.$$

Applying Bayes' theorem and the above formula we can evaluate the following posterior probabilities:

$$Pr(g/P) \quad = \quad \frac{Pr(P/g)Pr(g)}{Pr(P/g)Pr(g) + Pr(P/p)Pr(p)} \tag{7.5}$$

$$= \quad \frac{(0.7)(0.7)}{(0.7)(0.7) + (0.2)(0.3)} \tag{7.6}$$

$$= \quad 0.891 \tag{7.7}$$

$$Pr(p/P) \quad = \quad 0.109 \tag{7.8}$$

$$Pr(g/N) \quad = \quad \frac{Pr(N/g)Pr(g)}{Pr(N/g)Pr(g) + Pr(N/p)Pr(p)} \tag{7.9}$$

$$= \quad \frac{(0.3)(0.7)}{(0.3)(0.7) + (0.8)(0.3)} \tag{7.10}$$

$$= \quad 0.467 \tag{7.11}$$

$$Pr(p/N) \quad = \quad 0.533. \tag{7.12}$$

Now we can construct a new decision tree taking into account these posterior probabilities and determine how Mic can maximize his expected profit. The complete decision tree is shown in Figure 7.2.

This time the first node of the decision tree is chance node 1. The two branches, tagged Positive and Negative, emanating from node 1 represent the possible outcomes of the market report. The probabilities of occurence of these events are calculated as follows:

$$\begin{aligned}
Pr(Positive) = Pr(P) \quad &= \quad Pr(P/g)Pr(g) + Pr(P/p)Pr(p) \\
&= \quad (0.7)(0.7) + (0.2)(0.3) \\
&= \quad 0.49 + 0.06 = 0.55. \\
Pr(Negative) = Pr(N) \quad &= \quad Pr(N/g)Pr(g) + Pr(N/p)Pr(p) \\
&= \quad (0.3)(0.7) + (0.8)(0.3) \\
&= \quad 0.21 + 0.24 = 0.45.
\end{aligned}$$

The rest of the decision tree has the same structure as the tree in Figure 7.1. But now the previously computed posterior probabilities are used for the branches emanating from chance nodes. To determine the decision that will maximize Mic's expected profit, we work backwards from right to left. At each chance node we calculate the expected payoff (profit from expanding at for example node 4, corresponds to state nature Expand $EV(Expand) = (0.891)800000 + (0.109)500000 = 767300$. For decision node 2, we evaluate the expected value as follows

$$\begin{aligned}
\max\{EV(Expand), EV(\text{Status quo}), EV(Sell)\} \quad &= \quad \max\{767300, 1141950, 320000\} \\
&= \quad 1141950
\end{aligned}$$

and for decision node 3 we get 640100. The expected value for chance node 1 is computed using the expected profit values for node 2 and 3 and the posterior probabilities of the events "Positive Report" and "Negative Raport" and is equal to 916117.5=(0.55)1141850 + (0.45)640100. The optimal sequence of decisions can be obtained from the decision tree as follows:

- If the report is Positive, Mic needs to choose the action Status quo.

- If the report is Negative, Mic needs to choose the action Expand.

This strategy gives Mic his maximal expected profit of 916117.5. Without this additional information (posterior probability), Mic's expected profit equals \$865000.
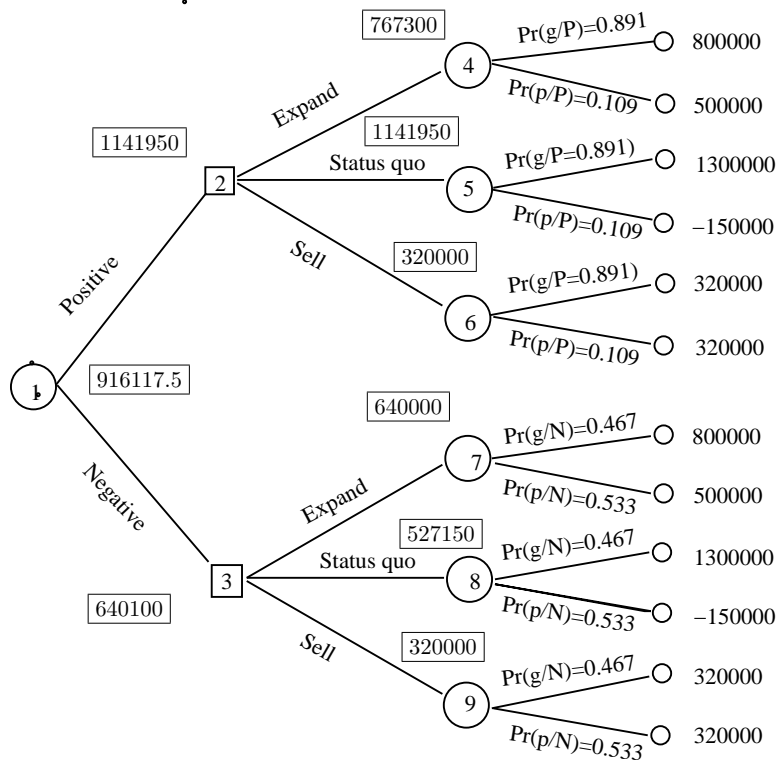
Figure 7.2: Decision tree with posterior probabilities

Chapter 8

Multiple Objective Linear Programming and Goal Programming

8.1 MULTIPLE OBJECTIVE LINEAR PROGRAMMING (MOLP)

Every conscious decision maker (DM) desires to realize his aims optimally. What does this mean in situations where the DM needs to consider multiple criteria(objectives) in arriving at an overall best decision. We now consider the situation in which the set of feasible decisions and set of q criteria (we assume that there are at least two criteria) for evaluating the decisions can be described in the form of the maximization of q linear objective functions subject to m linear constraints. Formally, a Multiple Objective Linear Programming Problem (MOLPP) can be described as:

$$
\begin{aligned}
f_1(x) = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n &\quad\rightarrow\quad maximize \\
f_2(x) = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n &\quad\rightarrow\quad maximize \\
\ddots\qquad \vdots\quad \vdots& \\
f_q(x) = c_{q1}x_1 + c_{q2}x_2 + \cdots + c_{qn}x_n &\quad\rightarrow\quad maximize
\end{aligned}
$$

subject to

$$
\mathbb{X}\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &=\quad b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &=\quad b_2 \\
\qquad\ddots \qquad\qquad\quad \vdots\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &=\quad b_m \\
\quad x_j \geq 0, j = 1, \ldots, n.
\end{cases}
$$

The set of feasible solutions $\mathbb{X} \subset \mathbb{R}^n$ is defined by the set of m constraints (inequalities or equations) for the problem and the nonnegativity constraints. Each feasible solution $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{X}$ (which is an n-dimensional vector of decision variables in the so called decision space $\mathbb{R}^n$) implies a value for each objective function $f_i(x), i = 1, \ldots, q$. Hence, each $\boldsymbol{x} \in \mathbb{X}$ corresponds to a vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_q) \in \mathbb{R}^q$, where

$$
\boldsymbol{y} = (y_1, y_2, \ldots, y_q) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_q(\boldsymbol{x})).
$$

The i-th coordinate of the vector $\boldsymbol{y}$ ($y_i = f_i(\boldsymbol{x})$) is an evaluation of the decision $\boldsymbol{x}$ from the point of view of the i-th criterion. The set of vectors $\boldsymbol{y}$ corresponding to the set $\mathbb{X}$ is denoted by $\mathbb{Y} \in R^q$:

$$\mathbb{Y} = \{\boldsymbol{y} = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_q(\boldsymbol{x})) : \boldsymbol{x} \in \mathbb{X}\}.$$

The set R^q is called objective space. If there exists a feasible solution $\bar{\bar{\boldsymbol{x}}}$ such that the maximum of each objective function occurs at the point $\bar{\bar{\boldsymbol{x}}}$, then this solution is optimal (it is a called *perfect solution*). But this happens very seldom - frequently such a point does not exist or does not belong to the set of feasible solutions $\mathbb{X}$. The concept of a **noniferior or Pareto-optimal solution or efficient solution** for a Multiple Objective Linear Programming Problem is defined as follows:

Definition 6. *A noninferior solution is a feasible solution to the problem, $\boldsymbol{x}^0 \in \mathbb{X}$, such that no other feasible solution, $\boldsymbol{x}$, exists for which $f_p(\boldsymbol{x}) > f_p(\boldsymbol{x}^0)$ for some $p = 1, 2, \ldots, q$, and $f_i(\boldsymbol{x}) \geq f_i(\boldsymbol{x}^0)$ for all $i \neq p$.*

Noninferior solutions can be defined in decision space ($\mathbb{X}$), as well as in objective space ($\mathbb{Y}$). We will denote by $\mathbb{X}_P$ the set of all noniferior (Pareto-optimal) solutions to the MOLPP in decision space, and by $\mathbb{Y}_P$ we denote the set of noninferior (also called nondominated) solutions in objective space. So we have:

$$\mathbb{Y}_P = \{\boldsymbol{y} = (y_1, y_2, \ldots, y_q) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_q(\boldsymbol{x})) : \boldsymbol{x} \in \mathbb{X}_P\}.$$

Now we present a fundamental insight into a property of the sets $\mathbb{X}_P$ and $\mathbb{Y}_P$.

Property 5. *If the set $\mathbb{X}$ is a bounded polyhedral set(polytope), then the set $\mathbb{Y}$ is also a polytope. Every extreme point (vertex) of $\mathbb{Y} = \{\boldsymbol{y} = (y_1, y_2, \ldots, y_q) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_q(\boldsymbol{x})) : \boldsymbol{x} \in \mathbb{X}\}$ is the projection of an extreme point of $\mathbb{X}$.*

Property 6. *Nondominated (noninferior) points in the set $\mathbb{Y}$ belong to the periphery of the set $\mathbb{Y}$. No internal point of $\mathbb{Y}$ can be a nondominated solution.*

Let us consider the following example with two criteria :

Example 15.

$$f_1(x) = -4x_1 + 3x_2 \quad \rightarrow \quad maximize$$
$$f_2(x) = 7x_1 + 5x_2 \quad \rightarrow \quad maximize$$

subject to

$$\mathbb{X} \begin{cases} x_1 & + & x_2 & \geq & 3 & (1) \\ -2x_1 & + & 3x_2 & \leq & 12 & (2) \\ & & x_2 & \leq & 6 & (3) \\ 6x_1 & + & x_2 & \leq & 42 & (4) \\ x_1 & & & \geq & 0 & (5) \\ & & x_2 & \geq & 0 & (6). \end{cases}$$

Using this example, we now illustrate the main notions of MOLPP.

We can see from Figure 8.1 that the set of feasible solutions $\mathbb{X}$ is the polygon ABCDEF, the objective functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ are represented by broken lines, $\bar{y}_1 = 12 = f_1(x_1, x_2)$ and $\bar{y}_2 = 72 = f_2(x_1, x_2)$. The perfect solution is the point $K(3\frac{33}{41}, 9\frac{3}{41})$ but it does not belong to the set of feasible solutions, so it cannot be a solution to the problem. First, to find the solution, i.e. the set $\mathbb{X}_P$, for this two-criteria problem , we will plot the set $\mathbb{Y}$. The extreme points (vertexes) of the set $\mathbb{X}$ and the values of the functions f_1, f_2 at these points are given in Table 8.1.
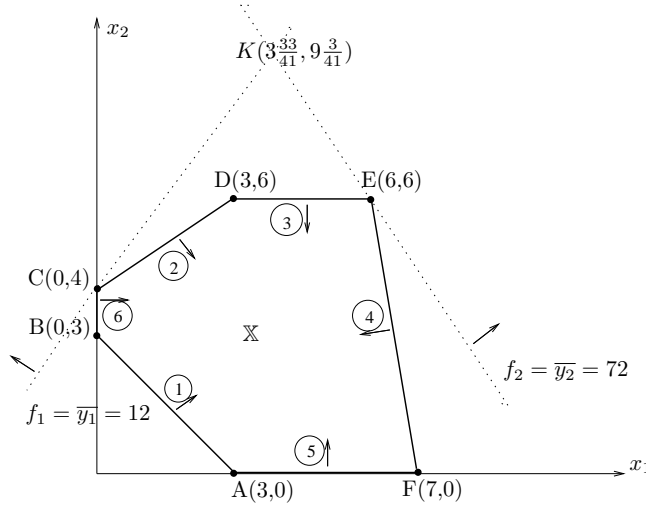
Figure 8.1: Decision space for the Two-Criteria Decision Problem

Node $\boldsymbol{x}$	$y_1 = f_1(\boldsymbol{x})$	$y_2 = f_2(\boldsymbol{x})$	$y(\boldsymbol{x}) = (y_1, y_2)$
A(3,0)	-12	21	y(A)=(-12,21)
B(0,3)	9	12	y(B)=(9,12)
C(0,4)	12	20	y(C)=(12,20)
D(3,6)	6	51	y(D)=(6,51)
E(6,6)	-6	72	y(E)=(-6,72)
F(7,0)	-28	49	y(F)=(-28,49)

Table 8.1: Extreme nodes of $\mathbb{X}$

The set $\mathbb{Y}$ is shown in Figure 8.2 as the polytope $y(A)y(B)y(C)y(D)y(E)y(F)$. Each vertex from $\mathbb{Y}$ has a corresponding extreme point in $\mathbb{X}$(Property 1). But the converse property is not valid in general. There could be an extreme point $\boldsymbol{x}' \in \mathbb{X}$ such that the point $y(\boldsymbol{x}')$ is not an extreme point in $\mathbb{Y}$ (this is not the case in the example considered). The set of nondominated solutions in $\mathbb{Y}$ is the set of points of edge $y(C)y(D)$ along the edge $y(D)y(E)$ (solid lines) of the polytope $y(A)y(B)y(C)y(D)y(E)y(F)$. Finally, because a nondominated point in $\mathbb{Y}$ corresponds to a noniferior feasible solution in $\mathbb{X}$, the solution to the two-criteria problem is the set of points along the edges CD and DE in the polytope ABCDEF (see Figure 8.1). A lot of techniques for solving MOLPP have been developed. Many of them are confined to the generating the set of noniferior solutions to the problem, and do not require any specification of the priority of goals by the decision maker (see Ralph E. Steur *Multiple Criteria Optimization: Theory, Computation, and Application*). Now we proceed to Goal Programming which requires such a specification.
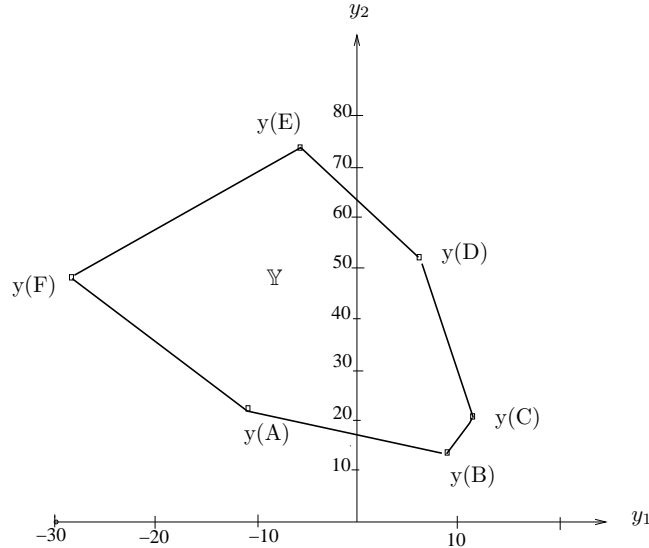
Figure 8.2: Objective space for the Two-Criteria Decision Problem

8.2   GOAL PROGRAMMING (Charnes and Cooper 1961)

Goal Programming(GP) is based upon minimizing the sum of the weighted absolute deviations of objectives $f_i(\boldsymbol{x})$ from targets T_i. Thus, the GP formulation of the initial MOILP is the following linear programming problem:

$$\min \sum_{i=1}^{q} w_i(d_i + e_i),$$

subject to

$$
\begin{aligned}
\boldsymbol{x} &\in \mathbb{X} \\
f_i(\boldsymbol{x}) - d_i + e_i &= T_i, \text{ for } i = 1, \ldots, q \\
d_i, e_i &\geq 0, \text{ for } i = 1, \ldots, q,
\end{aligned}
$$

where d_i and e_i are the positive and negative parts of the differences of the i-th objective from its target, respectively, and w_i is the weight or priority attached to the i-th goal. The w_i can also be split into two components - one for positive differences (d_i) and one for negative differences (e_i). We will use an example to illustrate goal programming.

Example 16. *A workshop is planning the monthly production of three products: A,B, and C. Product A requires 6 minutes, B 4 minutes and C 5 minutes to be fabricated. The workshop has 250 hours available each month, beyond which overtime is required, at a cost of $15 per hour. Painting the products is also required, with times of: 20 minutes per unit of A, 24 minutes per unit of B, and 30 minutes per unit of C. There are three painters in the workshop, each working 200 hours per*

month. The factory cannot sell a total of more than 3000 units in a month. The profit margins are $20, $28, and$40 for A,B, and C respectively. The workshop has four goals:

1. *A desired monthly profit level of $120000.*

2. *Overtime costs of $1000 is allowed.*

3. *Finishing hours should not exceed 1200 each month.*

4. *At least 200 of B should be produced.*

The formulation of the goal programming (GP) model of this problem is as follows. The decision variables:

$$
\begin{aligned}
x_1 &= \text{ number of units of A produced per month,} \\
x_2 &= \text{ number of units of B produced per month,} \\
x_3 &= \text{ number of units of C produced per month.}
\end{aligned}
$$

GP model:
$$Z = e_1 + d_2 + d_3 + e_4 \rightarrow minimize$$

subject to

$$
\begin{aligned}
x_1 + x_2 + x_3 &\leq 3000 \text{ (The system constraint)} & (8.1) \\
20x_1 + 28x_2 + 40x_3 + e_1 - d_1 &= 120000 \text{ (Profit)} & (8.2) \\
1.5x_1 + x_2 + 1.25x_3 + e_2 - d_2 &= 4750 \text{ (Overtime limit)} & (8.3) \\
20x_1 + 24x_2 + 30x_3 + e_3 - d_3 &= (1200 \times 60 =)72000 \text{ (Finishing time)} & (8.4) \\
x_2 + e_4 - d_4 &= 200 \text{ (Production of product B)} & (8.5) \\
x_i, d_j, e_j &\geq 0 \text{ for } i = 1,2,3; j = 1,\dots,4. & (8.6)
\end{aligned}
$$

The constraint 8.1 is only one restriction imposed on the levels of production and it is called the system constraint. Constraints 8.2 - 8.5 are called goal constraints because they define the deviation - d_i=the amount by which the i-th goal(target) is overachieved or e_i= the amount by which the i-th goal(target) is underachieved. Constraint 8.3 is obtained as follows. The demand for time (in minutes) is: $6x_1 + 4x_2 + 5x_3$. The supply is 250 hours×60 minutes=15000 minutes. Overtime will be paid after 15000 minutes, that is: Overtime=$(6x_1 + 4x_2 + 5x_3)$-15000. The cost of overtime is: $15 per hour or $0.25 per minute. Therefore we have: $0.25[(6x_1 + 4x_2 + 5x_3) - 15000] + d_2 - e_2 = 1000$ and finally we get constraint 8.3. There are only undesirable deviations in the objective function.

Let us assume that in the above example the manager could exactly determine the relative importance of the four goals. For example, suppose he determined that profit (goal 1) is the most important goal, the next is production of B (goal 4), overtime (goal 2) and painting time (goal 3) are both the least important goals. Weights for the arguments of the objective function must be introduced into the GP model. For example, suppose $w_1 = 5$, $w_4 = 3$, $w_2 = w_3 = 1$ (i.e. goal 1 is 5 times as important as the goal 2, goal 4 is 3 times as important as goal 2, goals 2 and 3 are equally the least important goals). Then the objective function of the GP model would be

$$Z = 5e_1 + d_2 + d_3 + 3e_4 \rightarrow minimize$$

In many situations, a decision maker may not be able to precisely determine the relative importance of the goals, but she/he can rank the goals from the most

important (goal 1) to the least important (goal q). When this the case, pre-emptive goal programming is used. The idea of the method is as follows. The decision maker first tries to satisfy the most important goal 1. Then, among all the points that satisfy goal 1, the decision maker tries to come as close as possible to satisfying goal 2, and so forth. We continue in this fashion until the only way we can come closer to satisfying a goal is to increase the deviation from a higher-priority goal. The interested reader can find more details in the texbook by Anderson, Sweeney and Williams *An Introduction to Management Science.*